STOCHASTIC PROGRAMMING

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Certificate

This is to certify that Mr. Saiful Islam Ansari has carried out the work reported in the present dissertation entitled "**Stochastic Programming**" under my supervision and that his dissertation work is suitable for submission for the degree of *Master of Philosophy* in *Statistics.*

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CONTENTS

PREFACE

The present dissertation entitled **"Stochastic programming"** is submitted to the Aligarh Muslim University, Aligarh, India, in partial fulfilment of the requirement for the award of degree of Master of Philosophy in Statistics. It consists of four chapters with comprehensive list of references, arranged in alphabetical order is also provided at the end of the dissertation. Each chapter is constructed so that the introductory and background material are presented first.

The development of various methods for the problem of Mathematical Programming in diverse field has been of primary concern of the Operations Analysts for last many decades. Mathematical Programming is concerned with Optimization problems of obtaining the best possible result under the circumstances. The result is measured in terms of an objective which is minimized or maximized. The circumstances are defined by a set of equality and/or inequality constraints.

Chapter I presents a short introductory discussion on mathematical programming and its various classifications both deterministic and stochastic programming. Further, the formulation approach, and brief introduction to solution methods are some of the subjects that have been covered in this chapter.

Chapter II presents two methods for solving the stochastic programming problem, namely, two stage stochastic programming and chance constrained programming. In chance constrained programming different cases also discussed when parameters are random variables. In the last section we mentioned some applications in various fields such as energy, finance, production, engineering, supply chain management, sports, catastrophe management and others.

Chapter III deals with two probabilistic models with Cauchy and extreme value distributions for stochastic programming. This chapter also presents probabilistic linear programming problem with joint constraints for Cauchy and extreme value distributions. Finally some numerical examples have also been presented to illustrate the methodology.

Chapter IV explains the cases when parameters of the probabilistic linear programming problem are considered as normal and log-normal random variables. A non-linear programming method has been used to solve the single-objective deterministic problem and a fuzzy programming method also used to solve the multi-objective deterministic problem. Finally a numerical example has also been presented to illustrate the methodology.

CHAPTER-I

INTRODUCTION

1.1 An Overview

Since the beginning of the history of mankind, man has been confronted with, and intrigued by the problem of deciding a course of action that would be the best for him under circumstances. This process of making optional judgment according to various criteria is known as the science of decision-making. Unfortunately, there was no scientific method for such an important class of problems until very recently. It is only in 1930s that a systematic approach to the decision problem started developing, mainly due to the 'New-Deal' in the United States and similar attempts in other parts of the world to curve the great economic depression prevailing throughout the world during this period. As a result during the 1940s, a new science began to emerge out.

About the same time, during World War II, the military management in the United Kingdom called upon a group of scientists from different disciplines to use their scientific knowledge for providing assistance to several strategic and logical war problems. The encouraging results achieved by the British scientist soon motivated the military management of the U.S.A. to start on similar activities. The methodology applied by these scientists to achieve their objectives was named as Operations Research (O.R.) because they were dealing with "research on military operations".

Operations Research is a branch of mathematical Sciences which is concerned with the application of scientific methods and techniques to decision-making problems and with establishing the best or optimal solutions. The systematic approach to decision making generally involves three closely interrelated stages. The first stage towards optimization is to express the desired benefits, required efforts and collecting the other relevant data, as a function of certain variables that may be called "decision variable". The second stage continues the process with an analysis of the mathematical model and selection of an appropriate numerical technique for finding the optimal solution. The third stage consists of finding an optimal solution, in most cases on a computer.

1.2 Mathematical Programming Problem

Mathematical Programming first arose in the field of economics where allocation problems had been a subject of long interest to economists. Von Neumann in the late 1930s and 1940s developed a linear model of an expanding economy. Leontief in 1951 showed a practical solution method for linear type problems when demonstrated his input-output model of an economy. These economic solution procedures did not provide optimal solution, but only a satisfying solution, given the model's linear constraints. In 1941, Hitchcock formulated and solved the transportation type problem, which was also accomplished by Koopmans in 1947. In 1942, Kantorovitch formulated but did not solve the transportation problem. In 1945, the economist G. J. Stigler formulated and solved the "minimum cost diet" problem. During World War II a group of researchers under the direction of Marshall K. Wood sought to solve allocation type problems for the United States Air Force. One of the members of this group, George B. Dantzig, formulated and devised a solution procedure in 1947 for Linear Programming (L.P.) type problems. This solution procedure, called the Simplex method, marked the beginning of the field of study called mathematical programming. During the 1950s other researchers such as David Gale, H.W. Kuhn and A.W. Tucker contributed to the theory of duality in LP. Others such as Charnes and Cooper contributed numerous LP applications illustrating the use of M.P in managerial decision-making.

A general Mathematical Programming Problem can be stated as following:

$$Max\ (or\ Min\ )\quad Z = f\left(X\right) \tag{1.2.1}$$

$$Sub.to\quad g_i(X) \leq or = or \geq b_i \quad \forall \quad i=1,2,\ldots,m \tag{1.2.2}$$

$$\text{and} \qquad X \geq 0 \tag{1.2.3}$$

where Z= value of the objective function which measures the effectiveness of the decision choice.

$g_i(X)$= set of i^{th} constraints.

X =unknown variables that are subject to the control of the decision maker.

b_i= available productive resources in limited supply.

The objective function is a mathematical equation describing a functional relationship between various decision variables and the outcome of the decisions. The outcome of managerial decision-making is the index of performance, and is generally measured by profits, sales, costs, or time. Thus, the value of the objective function in M.P. is expressed in monetary, physical, or some other terms, depending on the nature of the problem situation and of the decision to be made. The objective function may be either a linear or nonlinear function of variables. The objective of the decision maker is to select the values of the variables so as to optimize the value of the objective function Z frequently; the decision maker is confronted with making a sequence of interrelated decisions over time to optimize overall outcomes. This type of decision-making process is dynamic, rather than static.

1.3 Linear Programming Problem

Linear Programming (LP) is a mathematical technique most closely associated with operations research and management science. Linear programming is concerned with problems, in which a linear objective function in terms of decision variables is to be optimized (i.e., either minimized or maximized) while a set of linear equations, inequalities and sign restrictions are imposed on the decision variables as requirements (A linear equation/inequality is the one, which does not have a multi-degree polynomial within it). A linear programming problem is often referred to as an allocation problem because it deals with allocation of resources to alternative uses.

A general Linear Programming Problem can be described as follows:

$$Max(or\,Min) \quad Z = \sum_{j=1}^{n} c_j x_j \tag{1.3.1}$$

Subject to

$$\sum_{j=1}^{n} a_{ij}\, x_j \;\le or \;=\; or \;\ge b_i \quad \forall \quad i = 1,2,\ldots,m \tag{1.3.2}$$

and $x_j \ge 0 \quad \forall \quad j = 1,2,\ldots,n$ (1.3.3)

Linear Programs have turned out to be appropriate models for solving practical problems in many fields. G. B. Dantzig first conceived the linear programming problem in 1947. Koopman and Dantzig coined the name 'Linear Programming' in 1948, and Dantzig proposed an effective 'simplex method' for solving linear programming problems in 1949. Dantzig simplex method solves a linear program by examining the extreme points of a convex feasible region. Linear programming is often referred to as a Uni-objective constrained optimization technique. Uni-objective refers to the fact that linear programming problems seek to either maximize an objective such as profit or minimize the cost. The maximization of profit or minimization of cost is always constrained by the real world limitations of finite resources. LP allows decision makers an opportunity to combine the constraining limitations of the decision environment with the interaction of the variables they are seeking to optimize.

Development of new techniques for solving LPP is still going on. Decades of work on Dantzig's simplex method had failed to yield a polynomial-time variant. The first polynomial-time LP algorithm called Ellipsoid algorithm, developed by Khachiyan (1979), opened up the possibility that non-combinatorial methods might beat combinatorial one for linear programming. Karmarker (1984) developed a new polynomial time algorithm, which often outperform simplex method by a factor of 50 on real world problems. Some recent polynomial-time algorithms developed by Reneger (1988), Gonzaga (1989), Monteiro and Adler (1989), Vaidya (1990), Reha and Tutun (2000) are faster than Karmarkar's algorithm.

1.4 Non-Linear Programming Problem

Non-linear programming emerges as an increasingly important tool in economic studies and in operations research. Non linear programming problems arise in various disciplines as engineering, business administration, physical sciences and in mathematics or in any other area where decision must be taken in some complex situation that can be represented by a mathematical model:

$$\left.\begin{array}{l} \textit{Minimize } f(x) \\ \textit{Sub.to } \; g_i(x) \geq 0, \; i = 1,2,\ldots,m \\ \qquad\qquad x \geq 0 \end{array}\right\} \tag{1.4.1}$$

The functions $f(x)$ or $g(x)$ or both may be non linear function in x.

Interest in nonlinear programming problems developed simultaneously with the growing interest in linear programming. In the absence of general algorithms for nonlinear programming problems, it lies near at hand to explore the possibilities of approximate solution by linearization. The nonlinear functions of a mathematical programming problem were replaced by piecewise linear functions, these approximations may be expressed in such a way that the whole problem is turned into linear programming.

Kuhn & Tucker (1951) published an important paper "Nonlinear programming", dealing with necessary and sufficient conditions for optimal solutions to programming problems, which laid the foundations for a great deal of later work in nonlinear programming.

1.5 Multi-Objective Programming Problem

After the development of the simplex method by Dantzig for solving linear programming problems, various aspects of single objective mathematical programming have been studied quite extensively. It was however, realized that almost every real life problem involves more than one objective. Multi objective programming is a powerful mathematical procedure and applicable in decision making to a wide range of problems in the govt. Organizations, non -profitable organizations and private sector etc.

A multiple objective linear programming model with P objective functions can be stated as fallows:

$$\left.\begin{array}{ll} \textit{Max or Min} & \{f_1(X), f_2(X), \dots, f_p(X)\} \\ \textit{Sub. to} & X \in S \end{array}\right\} \quad (1.5.1)$$

where $f_i(X)$ $,\forall\, i = 1, 2, \dots, P$ is a linear function of the decision variable X and S is the set of feasible solutions. The ideal solution for a multiple objective linear programming problem would be to find that feasible set of decision variables X, which would optimize the individual objective functions of the problem simultaneously. However, with the conflicting objectives in the models, a feasible solution that optimizes one objective may not optimize any of the other remaining objective functions. This means that what is optimal in terms of one of the p objectives is generally not optimal for the other $P-1$ objectives i.e., multiple objective optimization has no way in which we may optimize all the objectives simultaneously. A number of methodologies have been developed to handle the problem of multiple objectives.

Methods of multi-objective optimization can be classified in many ways according to criteria. In Cohn (1985), they are categorized into two relatively distinct subsets: generating methods and preference-based method. In generating methods, the set of Pareto optimal (or efficient) solutions is generated for the decision maker, who then chooses one of the alternatives. In preference-based methods, the preferences of the decision maker are taken into consideration as the solution process goes on, and the solution that best satisfies the decision maker's preferences is selected.

Infact there is no universally accepted definition of "optimum" in multiple objective optimizations as in single objective optimization, which makes it difficult to even compare results of one method to another. Normally the decision about what the "best" answer is corresponds to the so-called human decision maker Coello (1999).

1.6 Goal Programming Problem

The Goal Programming (GP) is the most widely and suitable technique for solving the multi-objective linear problems. In searching for the origin of the goal programming analysis some analysts start with G.B. Dantzig's (1947) iterative procedure used in the analysis. While this start may be appropriate, it does not focus clearly on the specific nature of what is known today as goal programming. The ideas of goal programming were originally conceived by Charnes in (1955) for solving multi-objective linear programming problems. One of the most significant contributions that stimulated interest in the applications of GP was due to Charnes and Cooper in 1961. They introduced the concept of goal programming in connection with unsolvable linear programming problems (LPP). Additionally they pointed the issue of goal attainment and the value of goal programming in allowing for goals to be flexibility included in the model formulation. Another contribution during 1960s that had a significant impact on the formulation of the goal programming models and their application was contained in a text written by Ijiri in 1965. He explained the use of "preemptive priority factors" to treat multiple conflicting objectives in accordance with their importance in the objective function. Ijiri also suggested the "generalized inverse approach" and doing so, established goal programming as a distinct mathematical programming technique. Goal Programming is suitable for the situations where a satisfactory solution is sought rather than an optimal one that seeks the attainment of more than one goal. It attempts to achieve a satisfactory level in the attainment of multiple (often conflicting) objectives. Thus goal programming, like other multiple objective techniques is meant not for optimizing but for satisfying "as close as possible". Since there is no well-accepted Operations Research technique to find the optimum solution for multiple objective optimization problems, goal programming gives *a better representation of the actual* problem.

In general the Goal Programming model can be stated as follows:

$$Min.\ Z = \sum_{i=1}^{P} w_i\, P_k\, d_i \ \ (for\ k = 1,2,\ldots K) \qquad (1.6.2)$$

Subject to

$$\sum_{j=1}^{n} a_{ij}\, x_j + d_i^- - d_i^+ = b_i \ \ (for\ i = 1,2,\ldots,P) \qquad (1.6.2)$$

$$x_j,\ d_i^-,\ d_i^+ \geq 0 \ (for\ i = 1,2,\ldots P;\ j = 1,2,\ldots n) \qquad (1.6.3)$$

where the objective function minimizes Z, which is the sum of weighted deviational variables. P_k are the preemptive priority factors. The weight w is assessed for each i^{th} deviational variable and attached to each k^{th} priority factors. The objective function is minimized subject to P goal constraints where $a_{ij}'s$ are the coefficients for the decision variables $x_j's$. There are n decision variable in the model. The value b_i represents the right-hand-side for the goal constraint.

1.7 Fuzzy Programming Problem

The mathematical model for a multi-objective mathematical programming problem can be presented as follows:

$$Max{:}\ f_k(x) = f_k(x_1, x_2, \ldots, x_n), \quad k = 1,2,\ldots,K \qquad (1.7.1)$$

Subject to

$$g_i(x_1, x_2, ..., x_n) \leq b_i,\ i = 1,2,...,m \tag{1.7.2}$$

$$x_j \geq 0,\quad j = 1,2,...,m. \tag{1.7.3}$$

It is assumed that the functions $f_k(x), k = 1,2,...,K$, and $g_i(x),\ i = 1,2,...,m$ are of either the convex or the concave type (they may be linear or non-linear). The above problem can be described as a vector-maximum problem. We further assume that the problem is feasible and that there an optimal compromise exists. In chapter four we use Zimmerman's fuzzy programming technique [Zimmermann (1978), (1991)] to solve the problem. Fuzzy set theory for decision-making was first introduced by Bellman and Zadeh (1970). This technique has been applied to almost all mathematical programming problems, including linear programming, non-linear programming, stochastic programming and dynamic programming, and to many other real-life mathematical programming problems [Kibzun and Kan (1996); Mohan and Nguyen (1997); Romero (2004); Ballestero (2001)]. In this section we present a brief fuzzy programming method for solving the deterministic problem.

Let $X_1^{(1)}, X_2^{(2)}, ..., X_k^{(k)}$ be the ideal solutions for the respective objective function. Using the above ideal solutions, we formulate a pay-off matrix. Then lower and upper bound of each of the objective functions is estimated from the pay-off matrix as

$$L_k \le f_k \le U_k, \; k = 1,2,...,K. \tag{1.7.4}$$

Next, we define a fuzzy membership function for the k^{th} objective function f_k:

$$\mu_{f_k}(x) = \begin{cases} 1 & if \quad f_k \ge U_k \\ 1 - \dfrac{(U_k - f_k)}{(U_k - L_k)} & if \quad L_k < f_k < U_k \\ 0 & if \quad f_k \le L_k \end{cases} \tag{1.7.5}$$

The above membership function is used to formulate a crisp model:

$$Min : \lambda \tag{1.7.6}$$

Subject to

$$f_k(x_1, x_2, ..., x_n) + (U_k - L_k) \ge U_k, \quad k = 1,2,...,K \tag{1.7.7}$$

$$g_i(x_1, x_2, ..., x_n) \le b_i, \quad i = 1,2,...,m \tag{1.7.8}$$

$$\lambda \ge 0, \; x_j \ge 0, \; j = 1,2,...,n. \qquad (1.7.8)$$

1.8 Stochastic Programming Problem

Stochastic programming is a framework for modeling optimization problems that involve uncertainty. Whereas deterministic optimization problems are formulated with known parameters, real world problems almost invariably include some unknown parameters. When the parameters are known only within certain bounds, one approach of tackling such problem is called robust optimization. Here is a goal to find a solution, which is feasible for all such data and optimal in some sense. Stochastic programming models are similar in style but take advantage of the fact that probability distributions governing the data are known or can be estimated. The goal here is to find some policy that is feasible for all (or almost all) the possible data, for instances for maximize the expectation of some function of the decisions and random variables. More generally, such models are formulated, solved analytically or numerically, and analyzed in order to provide useful information to a decision maker.

Beginning with the seminal work of Beale (1955) Bellman (1957), Belmam and Zadeh (1970), Charnes and Cooper (1959), Dantzig (1955) and Tintner (1955), optimization under uncertainty has experienced rapid development in both theory and algorithms. For detail information related to stochastic optimization there are many recent text books of Bertsekas and Tsitsiklis (1996), Birge and Louveaux (1997), Kall and Wallace (1994), Pre'kopa (1998) and Zimumermann (1991) and a very informative stochastic programming community home page.

A Stochastic linear programming problem can be stated as:

$$Maximize f(x) = \sum_{j=1}^{n} c_j x_j \tag{1.8.1}$$

$$\text{Subject to} \quad \sum_{j=1}^{n} a_{ij} x_j \geq b_i \tag{1.8.2}$$

and $\quad x_j \geq 0; \quad j = 1,2,\ldots,n.$ (1.8.3)

where some of all the coefficients c_j, a_{ij} and b_i are random variables.

(a) Two Stage Stochastic Programming

Two-stage stochastic programming is concerned with problems that require a here-and-now decision on the basis of given probabilistic information on the random data without making further observations. The costs to be minimized consist of the direct costs of the here-and now (or first-stage) decision as well as the costs generated by the need of taking a recourse (or second-stage) decision in response to the random environment. Recourse costs are often formulated by means of expected values with respect to the probability distribution of the involved random data. In this way, two-stage models and their solutions depend on the underlying probability distribution. Since this distribution I sís often incompletely known in applied models, or it has to be approximated for computational purposes, the stability behaviour of stochastic programming models when changing the probability measure is important. This problem is studied in a number of papers published by Artstein and Wets (1990), King and Rockafellar (1993), Romisch and Schultz (1993, 1996), Shapiro (1990, 1991). Artstein and Wets (1990) obtained general results on continuity properties of optimal values and solutions when perturbing the probability measures with respect to the topology of weak convergence. Quantitative continuity results of solution sets to two-stage stochastic programs with respect to suitable distances of probability measures are obtained by Römisch and Schultz (1993, 1996). Asymptotic properties of statistical estimators of values and solutions to stochastic programs are derived King and Rockafellar (1993) and Shapiro (1990, 1991).

(*b*) *Chance Constrained Programming*

Although two-stage stochastic linear programs are often regarded as the classical stochastic programming-modeling paradigm, the discipline of stochastic programming has grown and broadened to cover a wide range of models and solution approaches. Applications are widespread, from finance to fisheries management. An alternative modeling approaches used so-called Chance constraints. These do not require that our decisions are feasible for (almost) every outcome of the random parameters, but require feasibility with at least some specified probability. One natural generalization of two stage model extends it to many stages. Here each stage consists of a decision followed by a set of observations of the uncertain parameters which are gradually revealed overtime. In this context stochastic programming is closely related to decision analysis, optimization of discrete event simulations, stochastic control theory, Markov decision process, and dynamic programming.

In chance constrained programming, the stochastic linear programming problem is stated as follows:

$$\text{Minimize} \quad f(X) = \sum_{j=1}^{n} c_j x_j \tag{1.10.1}$$

Subject to

$$P\left[\sum_{j=1}^{n} a_{ij} x_j \le b_i\right] \ge p_i, \quad i = 1,2,\ldots,m \tag{1.10.2}$$

$$\text{and} \quad x_j \ge 0; \; j = 1,2,\ldots,n \tag{1.10.3}$$

where, c_j, a_{ij} and b_i are random variables and p_i are specified probabilities.

Chance constrained programming was formulated originally by Charnes Cooper and Symonds (1958) and Charnes and Cooper (1959) and has since been further developed and applied by Charnes and Cooper (1962, 1963), Charnes Cooper and Thompson (1964, 1965), Bel Israel (1962), Kataoka (1963), Kirby (1965), Naslund (1966), Naslund and Whinston (1962), Thiel (1961), Van De Panve and Popp (1963) and Miller and Wagner (1965).

CHAPTER II

STOCHASTIC PROGRAMMING: METHODS AND APPLICATIONS

2.1 Introduction

Stochastic programming is an approach for modeling optimization problems that involve uncertainty. Whereas deterministic optimization problems are formulated with known parameters, real world problems almost invariably include parameters which are unknown at the time a decision should be made. When the parameters are uncertain, but assumed to lie in some given set of possible values, one might seek a solution that is feasible for all possible parameter choices and optimizes a given objective function. Such an approach might make sense for example when designing a least-weight bridge with steel having a tensile strength that is known only to within some tolerance. Stochastic programming models are similar in style but try to take advantage of the fact that probability distributions governing the data made repeatedly in essentially the same circumstances, and the objective is to come up with a decision that will perform well on average. An example would be designing truck routes for daily milk delivery to customers with random demand. Here probability distributions (e.g., of demand) could be estimated from data that have been collected over time. The goal is to find some polity that is feasible for all (or almost all) the possible parameter a realization and optimizes the expectation of some function of the decisions and the random variables.

Stochastic optimization problem have been studied since the work of Dantzig (1955) and Beale (1955) in the 1950's and attempt to model uncertainty in the data by assuming that (part of) the input is specified in terms of a probability distribution rather than by the deterministic data given in advance. Since the work of Dantzig, Stochastic optimization, also referred to as stochastic programming, has grown into a tremendous field with a vast literature including various text books as Kall and Wallace (1994), Pre'kopa (1995) and Birge and Louveaux (1997).

A Stochastic linear programming problem can be stated as:

$$\textit{Maximize } f(X) = C^T X = \sum_{j=1}^{n} c_j x_j \tag{2.1.1}$$

$$\text{Subject to } A_i^T X = \sum_{j=1}^{n} a_{ij} x_j \geq b_i \tag{2.1.2}$$

$$\text{and} \qquad x_j \geq 0;\ j = 1,2,\ldots,n \tag{2.1.3}$$

where some or all the coefficients c_j, a_{ij} and b_i are random variables with known probability distribution. The decision variables x_j are assumed to be deterministic for simplicity. Several methods are available for solving the problem stated in equations (2.1.1) to (2.1.3). However only two methods, namely, the two stage stochastic programming technique and the chance-constrained programming technique are discussed here.

2.2 Two Stage Stochastic Programming

The most widely applied and studied stochastic programming models are two stage linear programs. Where the decision maker takes some action in the first stage, after which a random event occurs affecting the outcome of the first stage decision. A recourse decision can then be made in the second stage that compensates for any bad effects that might have been experienced as a result of a first stage decision. The optimal policy from such a model is a single first stage policy and a collection of recourse decisions (a decision rule) defining which second stage action should be taken in response to each random outcome.

The two-stage programming technique is one, which converts a stochastic linear programming problem into an equivalent deterministic problem. This is accomplished at the expanse of increasing the size of the problem. For simplicity, we assume that only the elements b_i, are probabilistic. This means that the variable b_i is not precisely known, but its probability distribution function, with a finite mean $\bar{b}_i$,known to us. In this case, it is impossible to find a vector X in such a way that $A_i^T X$ will be greater than or equal to b_i $(i = 1,2,\dots,m)$ for whatever the value b_i takes. In fact, the difference between $A_i^T X$ and b_i

will itself be a random variable, whose probability distribution function depends on the value of, X chosen.

One can now think of associating a penalty for violation, we might get for the constraints. In this case, we can think of minimizing the sum of $C^T X$ and the expected value of the penalty. One choice is to assume a constant penalty cost of p_i for violating the i^{th} constraint by one unit.

Thus, the total penalty is given by the expected (mean) value of the sum of the individual penalties, $\sum_{i=1}^{m} E(p_i y_i)$ where E is the expectation and y_i is defined as

$$y_i = b_i - A_i^T X, \quad y_i \ge 0, \quad i = 1,2,\dots,m \tag{2.2.1}$$

Hence, we can add the mean total penalty cost to the original objective function and write the new optimization problem as:

Minimize $\quad C^T X + E(P^T Y) \quad$ (2.2.2)

Subject to $\quad AX + BY = b \quad$ (2.2.3)

and $\quad X \ge 0, Y \ge 0 \quad$ (2.2.4)

where $\quad P = \begin{Bmatrix} p_1 \\ p_2 \\ \vdots \\ p_m \end{Bmatrix}, \quad Y = \begin{Bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{Bmatrix}$

And $B = I$ = Identity matrix of order m.

Notice that penalty term in equation (2.2.2) will be deterministic quantity in terms of the expected values of $y_i, \bar{y}_i$.

To convert the problem stated in Equations (2.2.1) to (2.2.4) to a fully deterministic one, the probabilistic constraints, Equation (2.2.3), have to be written either in a deterministic form like $\bar{y}_i = \bar{b}_i - A_i^T X$ or interpreted as a two-stage problem as follows:

First Stage: First estimate or guess the vector b and find the vector X by solving the problem stated in Equations (2.1.1) to (2.1.3).

Second Stage: Then observe the value of $b$and hence its discrepancy from the previous guess vector, and find the vector $Y = Y(b, X)$ by solving the second stage problem:

$$\left.\begin{array}{l} \text{Find } Y \text{ which minimizes } P^T Y \\ \text{Subject to} \\ \qquad y_i = b_i - A_i^T X, \quad i = 1,2,\ldots,m \\ \text{and} \quad y_i \geq 0, \quad i = 1,2,\ldots,m \end{array}\right\} \tag{2.2.5}$$

where $b_i$and $X$are known now.

Thus, the two-stage formulation can be interpreted to mean that a non-negative vector X must be found (here and now) before the actual values of b_i $(i = 1,2,\ldots m)$ are known, and that when they are known, a recourse Y must be found by solving the second stage problem of equation (2.2.5). Hence, a general two-stage problem can be stated as follows:

$$\left.\begin{array}{ll} \text{Minimize} & C^T X + E\left[\min_{y}(P^T Y)\right] \\ \text{Subject to} & \underset{m\times n_1}{A}\ \underset{n\times 1}{X} + \underset{m\times n_2}{B}\ \underset{n_2\times 1}{Y} \geq \underset{m\times 1}{b} \\ \text{and} & X \geq 0, Y \geq 0 \end{array}\right\} \qquad (2.2.6)$$

where b is a random m dimensional vector with known probability distribution $F(b)$ and probability density function $dF(b) = f(b)$. The following assumptions are generally made to solve this problem.

(i) The penalty cost vector P is a known deterministic vector, and

(ii) There exists a nonempty convex set S consisting a nonnegative solution vector X such for each b, there exist a solution vector $Y(b)$ so that the pair $[X, Y(b)]$ is feasible.

The second assumption is called the assumption of permanent feasibility. By defining,

$$\underset{m\times(n_1+n_2)}{D} = [A, B] \qquad (2.2.7)$$

$$\underset{(n_1+n_2)\times 1}{Q} = \begin{Bmatrix} C \\ P \end{Bmatrix} \qquad (2.2.8)$$

$$\text{and} \qquad \underset{(n_1+n_2)\times 1}{Z}(b) = \begin{Bmatrix} X \\ Y(b) \end{Bmatrix} \tag{2.2.9}$$

The two-stage problem states in equation (2.2.6) can be expressed as:

$$\left.\begin{aligned} &\text{Minimize} \\ &\qquad \int Q^T Z(b) f(b) = \text{Expected cost} \\ &\text{Subject to} \quad DZ(b) \geq b \\ &\text{and} \qquad\quad Z(b) \geq 0 \quad \text{for all } b \end{aligned}\right\} \tag{2.2.10}$$

2.3 Chance Constrained Programming Technique

As the name indicates, chance constrained programming technique is one which can be used to solve problems involving chance constraints, that is, constraints having finite probability of being violated. This chance constrained programming permits the constraints to be violated by a specified (small) probability whereas the two-stage programming does not permit any constraint to be violated.

The chance constrained programming technique was originally developed by Charnes and Cooper (1959) and extended by Van De Panne and Popp (1963), Miller and Wagner (1965). Chance constrained models arises when exact values of the parameters are not known such as the nutritive contents of cattle feed problem discussed by Van De Panne and Popp (1963).

In chance constrained programming, the stochastic linear programming problem is stated as follows:

$$\text{Minimize} \quad f(X) = \sum_{j=1}^{n} c_j x_j \tag{2.3.1}$$

Subject to

$$P\left[\sum_{j=1}^{n} a_{ij} x_j \leq b_i\right] \geq p_i, \quad i = 1,2,\ldots,m \tag{2.3.2}$$

$$\text{and} \qquad x_j \geq 0; \ j = 1,2,\ldots,n \tag{2.3.3}$$

where c_j, a_{ij} and b_i are random variables and p_i are specified probabilities. Notice that Equation (2.3.2) indicate that the i^{th} constraint,

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i \tag{2.3.4}$$

has to be satisfied with a probability of at least p_i where $0 \leq p_i \leq 1$. For simplicity, we are assuming that the decision variables x_j are deterministic. We shall first consider special cases where only c_j or a_{ij} or b_i are random variables before considering the general case in which c_j, a_{ij} and b_i are all random variables. We shall further assume that all the random variables are normally distributed with known mean and standard deviations.

(i) When only a_{ij} are random variables: Let $\bar{a}_{ij}$ and $Var(a_{ij}) = \sigma^2_{a_{ij}}$ be the mean and the variance of the normally distributed random variable a_{ij}. Assume that the multivariate distribution of a_{ij}, $j = 1,2,...n$ is also known along with the covariance, $Cov(a_{ij}, a_{kl})$ between the random variables a_{ij} and a_{kl}. Define quantities d_i as

$$d_i = \sum_{j=1}^{n} a_{ij} x_j,\ i = 1,2,...,m \tag{2.3.5}$$

Since $a_{i1}, a_{i2},..., a_{in}$ are normally distributed, and $x_1, x_2,..., x_n$ are constants (not yet known), d_i will also be normally distributed with a mean value of

$$\bar{d}_i = \sum_{j=1}^{n} \bar{a}_{ij} x_j,\ i = 1,2,...,m \tag{2.3.6}$$

and a variance of

$$Var(d_i) = \sigma^2_{d_i} = X^T V_i X \tag{2.3.7}$$

where V_i is the i^{th} covariance matrix defined as

$$V_i = \begin{bmatrix} Var(a_{i1}) & Cov(a_{i1}, a_{i2}) & .... & Cov(a_{i1}, a_{in}) \\ Cov(a_{i2}, a_{i1}) & Var(a_{i2}) & .... & Cov(a_{i2}, a_{in}) \\ \vdots & \vdots & & \vdots \\ Cov(a_{in}, a_{i1}) & Cov(a_{i1}, a_{i2}) & .... & Var(a_{in}) \end{bmatrix} \tag{2.3.8}$$

The constraints of Equation (2.3.2) can be expressed as

$$P[d_i \le b_i] \ge p_i$$

i.e.,
$$P\left[\frac{d_i - \bar{d}_i}{\sqrt{Var(d_i)}} \le \frac{b_i - \bar{d}_i}{\sqrt{Var(d_i)}}\right] \ge p_i,\ i = 1,2,\ldots,m \tag{2.3.9}$$

where $\left[\left(d_i - \bar{d}_i\right)/\sqrt{Var(d_i)}\right] \sim N(0,1)$. Thus the probability of realizing d_i smaller than or equal to b_i can be written as

$$P[d_i \le b_i] = \phi\left(\frac{b_i - \bar{d}_i}{\sqrt{VAr(d_i)}}\right) \tag{2.3.10}$$

where $\phi(x)$ represents the cumulative distribution function of the standard normal distribution evaluated at x. If e_i denotes the value of the standard normal variable at which

$$\phi(e_i) = p_i \tag{2.3.11}$$

Then the constraints in Equation (2.3.9) can be stated as

$$\phi\left(\frac{b_i - \bar{d}_i}{\sqrt{Var(d_i)}}\right) \ge \phi(e_i),\ i = 1,2,\ldots,m \tag{2.3.12}$$

These inequalities will be satisfied only if

$$\left(\frac{b_i - \bar{d}_i}{\sqrt{Var(d_i)}}\right) \ge e_i$$

or ,
$$\bar{d}_i + e_i\sqrt{Var(d_i)} - b_i \le 0;\ i = 1,2,\ldots,m \tag{2.3.13}$$

By substitution Equations (2.3.6) and (2.3.7) in Equation (2.3.13), we obtain

$$\sum_{j=1}^{n} \bar{a}_{ij} x_j + e_i \sqrt{X^T V_i X} - b_i \le 0; \; i = 1,2,\ldots,m \qquad (2.3.14)$$

These are the deterministic nonlinear constraints equivalent to the original stochastic linear constraints.

Thus the solution of the stochastic programming problem stated in Equations (2.3.1) to (2.3.3) can be obtained by solving the equivalent deterministic programming problem:

$$\left.\begin{aligned} &\text{Minimize} \quad f(X) = \sum_{j=1}^{n} c_j x_j \\ &\text{Subject to} \\ &\quad \sum_{j=1}^{n} \bar{a}_{ij} x_j + e_i \sqrt{X^T V_i X} - b_i \le 0; \; i = 1,2,\ldots,m \\ &\text{and} \qquad x_j \ge 0, \; j = 1,2,\ldots,n \end{aligned}\right\} \qquad (2.3.15)$$

If the normally distributed random variables a_{ij} are independent the covariance terms will be zero and Equation (2.3.8) reduces to a diagonal matrix as

$$V_i = \begin{bmatrix} Var(a_{i1}) & 0 & \ldots & 0 \\ 0 & Var(a_{i2}) & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & Var(a_{in}) \end{bmatrix} \qquad (2.3.16)$$

In this case, the constraints of Equation (2.3.14) reduce to

$$\sum_{j=1}^{n} \bar{a}_{ij} x_j + e \sqrt{\sum_{j=1}^{n} \left[Var(a_{ij}) x_j^2 \right]} - b_i \leq 0 \qquad (2.3.17)$$

(ii) When only b_i are random variables: Let $\bar{b}_i$ and $Var(b_i)$ denote the mean and variance of the normally distributed random variable b_i. The constraints of Equation (2.3.2) can be restated as

$$P\left[\frac{b_i - \bar{b}_i}{\sqrt{Var(b_i)}} \geq \frac{\sum_{j=1}^{n} a_{ij} x_j - \bar{b}_i}{\sqrt{Var(b_i)}} \right] \geq p_i; \quad i = 1,2,\ldots,m \qquad (2.3.18)$$

where $\left[(b_i - \bar{b}_i) / \sqrt{Var(b_i)} \right] \sim N(0,1)$. The inequalities (2.3.18) can also be stated as:

$$P\left[\frac{b_i - \bar{b}_i}{\sqrt{Var(b_i)}} \leq \frac{\sum_{j=1}^{n} a_{ij} x_j - \bar{b}_i}{\sqrt{Var(b_i)}} \right] \leq 1 - p_i; \quad i = 1,2,\ldots,m \qquad (2.3.19)$$

If E_i represents the value of the standard normal variate at which

$$\phi(E_i) = 1 - p_i,$$

The constraints in Equation (2.3.19) can be expressed as

$$\phi\left(\frac{\sum_{j=1}^{n} a_{ij}x_j - \bar{b}_i}{\sqrt{Var(b_i)}}\right) \leq \phi(E_i),\ i = 1,2,...,m \tag{2.3.20}$$

These inequalities will be satisfied only if

$$\frac{\sum_{j=1}^{n} a_{ij}x_j - \bar{b}_i}{\sqrt{VAr(b_i)}} \leq E_i,\ i = 1,2,...,m$$

or,

$$\sum_{j=1}^{n} a_{ij}x_j - \bar{b}_i - E_i\ \sqrt{Var(b_i)} \leq 0,\ i = 1,2,...,m \tag{2.3.21}$$

Thus the stochastic linear programming problem stated in Equations (2.3.1) to (2.3.3) is equivalent to the following deterministic LPP:

$$\text{Minimize}\quad f(X) = \sum_{j=1}^{n} c_j x_j$$

$$\left.\begin{array}{l} \text{Subject to} \\ \sum_{j=1}^{n} a_{ij}x_j - \bar{b}_i - E_i\ \sqrt{Var(b_i)} \leq 0,\ i = 1,2,...,m \\ \text{and} \qquad x_j \geq 0\ ,\quad j = 1,2,...n. \end{array}\right\} \tag{2.3.22}$$

(iii) When only c_j are random variables: Since c_j are normally distributed random variables, the objective function $f(X)$ will also be a normally distributed random variable. The mean and variance of f are given by

$$\bar{f} = \sum_{j=1}^{n} \bar{c}_j x_j \tag{2.3.23}$$

and
$$Var\,(f) = X^T V X \tag{2.3.24}$$

where $\bar{c}_j$ is the mean value of c_j and the matrix V is the covariance matrix of c_j defined as

$$V = \begin{bmatrix} Var\,(c_1) & Cov\,(c_1,c_2) & \ldots & Cov\,(c_1,c_n) \\ Cov\,(c_2,c_1) & Var\,(c_2) & \ldots & Cov\,(c_2,c_n) \\ \vdots & \vdots & & \vdots \\ Cov\,(c_n,c_1) & Cov\,(c_n,c_2) & \ldots & Var\,(c_n) \end{bmatrix} \tag{2.3.25}$$

with $Var(c_j)$ and $Cov(c_i,c_j)$ denoting the variance of c_j and covariance between c_i and c_j respectively.

A new deterministic objective function for minimization can be formulated as

$$F(X) = k_1 \bar{f} + k_2 \sqrt{Var(f)} \tag{2.3.26}$$

where k_1 and k_2 are non negative constants whose values indicate the relative importance of $\bar{f}$ and standard deviation of f for, minimization. Thus $k_2 = 0$ indicates that the expected value of f is to be minimized without caring for the standard deviation of f. On the other hand, if $k_1 = 0$, it indicates that we are interested in minimizing the variability of f about its mean value without bothering about what happens to the mean value of f. Similarly, if $k_1 = k_2 = 1$, it indicates that we are giving equal importance to the minimization of the mean as well as the standard deviation of f. Notice that the new objective function stated in Equation (2.3.26) is a nonlinear function in X in view of the expression for the variance of f.

Thus the solution of the stochastic linear programming problem stated in Equations (2.3.1) to (2.3.3) can be obtained by solving the equivalent deterministic nonlinear programming problem:

$$\left.\begin{aligned}&\text{Minimize}\\ &F(X) = k_1 \sum_{j=1}^{n} \bar{c}_j x_j + k_2 \sqrt{X^T V X}\\ &\text{Subject to}\\ &\sum_{j=1}^{n} a_{ij} x_j - b_i \le 0,\; i = 1,2,\ldots,m\\ &\text{and}\quad x_j \ge 0,\; j = 1,2,\ldots,n\end{aligned}\right\}\tag{2.3.27}$$

If all the random variables c_j are independent, the objective function reduced to

$$F(X) = k_1 \sum_{j=1}^{n} \bar{c}_j x_j + k_2 \sqrt{\sum_{j=1}^{n} Var(c_j) x_j^2} \qquad (2.3.28)$$

(iv) When c_j, a_{ij} and b_i are random variables: As the random variables c_j, $j = 1,2,...,n$ appear only in the objective function, we can take the new objective function $F(X)$ same as the one given in Equation (2.3.26).

The constraints of Equation (2.3.2) can be expressed as

$$P[h_i \le 0] \ge p_i, i = 1,2,...,m \qquad (2.3.29)$$

where h_i is a new random variable defined as

$$h_i = \sum_{j=1}^{n} a_{ij} x_j - b_i = \sum_{k=1}^{n+1} q_{ik} y_k \qquad (2.3.30)$$

where $\quad q_{ik} = a_{ik}, k = 1,2,....,n$

$$q_{i,n+1} = b_i$$
$$y_k = x_k, k = 1,2,...,n,$$

and $\quad y_{n+1} = -1.$

Notice that the constant y_{n+1} is introduced for convenience. Since h_i is given by a linear combination of the normally distributed random variables q_{ik}. It will also follow normal distribution. The mean and the variance of h_i are given by

$$\bar{h}_i = \sum_{k=1}^{n+1} \bar{q}_{ik} y_k = \sum_{j=1}^{n} \bar{a}_{ij} x_j - \bar{b}_i \qquad (2.3.31)$$

and $\quad Var(h_i) = Y^T V_i Y \qquad (2.3.32)$

where $\quad Y = \begin{Bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n+1} \end{Bmatrix}$

and

$$V_i = \begin{bmatrix} Var(q_{i1}) & Cov(q_{i1}, q_{i2}) & \dots & Cov(q_{i1}, q_{i,n+1}) \\ Cov(q_{i2}, q_{i1}) & Var(q_{i2}) & \dots & Cov(q_{i2}, q_{i,n+1}) \\ \vdots & \vdots & & \vdots \\ Cov(q_{i,n+1}, q_{i1}) & Cov(q_{i,n+1}, q_{i2}) & \dots & Var(q_{i,n+1}) \end{bmatrix} \qquad (2.3.33)$$

Thus the constraints in Equation (2.3.29) can be restated as

$$P\left[\frac{h_i - \bar{h}_i}{\sqrt{Var(h_i)}} \le \frac{-\bar{h}_i}{\sqrt{Var(h_i)}}\right] \ge p_i, \; i = 1,2,\dots,m \qquad (2.3.34)$$

where $\left[(h_i - \bar{h}_i)/\sqrt{Var(h_i)}\right] \sim N(0,1)$.

Thus if e_i denotes the value of the standard normal variable at which

$$\phi(e_i) = p_i, \qquad (2.3.35)$$

The constraints of Equation (2.3.34) can be stated as

$$\phi\left(\frac{-\bar{h}_i}{\sqrt{Var(h_i)}}\right) \ge \phi(e_i), \; i = 1,2,\dots,m \qquad (2.3.36)$$

These inequalities will be satisfied only if the following deterministic nonlinear inequalities are satisfied:

$$\frac{-\bar{h}_i}{\sqrt{Var(h_i)}} \geq e_i,\ i = 1,2,\ldots,m$$

or $$\bar{h}_i + e_i\sqrt{Var\,(h_i)} \leq 0,\ i = 1,2,\ldots,m \tag{2.3.37}$$

Thus the stochastic linear programming problem of Equations (2.3.1) to (2.3.3) can be stated as an equivalent deterministic nonlinear programming problem as:

$$\left.\begin{aligned}
&\text{Minimize}\\
&f(X) = k_1\sum_{j=1}^{n}\bar{c}_j x_j + k_2\sqrt{X^T V X}, \quad k_1, k_2 \geq 0\\
&\text{Subject to}\\
&\bar{h}_1 + e_i\sqrt{Var(h_i)} \leq 0,\ i = 1,2,\ldots,m,\\
&\text{and} \quad x_j \geq 0,\ j = 1,2,\ldots,n
\end{aligned}\right\} \tag{2.3.38}$$

2.4Applications of Stochastic Programming

(*i*) Energy

A particularly important field of application of stochastic programming is the optimization of production, trading, storage, and transportation of all kinds of energy, i.e., electricity (power), gas, oil, etc.; see [Wallace and Fleten (2003)] for a recent survey. Typically, the stochastic nature of prices and demands cannot be neglected in energy optimization models. Especially the optimization of electricity production and trading (electricity portfolio management) seems to fit exceedingly well to the stochastic programming paradigm. One reason for this is that regulations for electricity trading include a fixed time discretization into intervals of, e.g., one hour length. Moreover, electricity is a non-storable commodity and, therefore, the consideration of the stochastic nature of the parameters becomes even more important since discrepancies at one time cannot be compensated at another time.

There is a lot of literature dealing with optimal power planning in terms of stochastic programming. A general distinction may be drawn between models for systems in regulated and in liberalized markets. However, several other distinctions can be made, e.g., with respect to the level of abstraction from physical aspects of electricity production and transmission. The classical application in regulated markets is the so-called unit commitment problem where a number of power production units (e.g., blocks of thermal power plants or hydroelectric power plants) has to be scheduled in such a way that the (Expected) fuel costs are minimized under the constraint that a (stochastic) demand of electricity is always met. In addition, there are technical constraints for each unit; see, e.g., [Nowak (2000); Nowak and Römisch, (2000); Gröwe-Kuska and Römisch, (2005)] for a seminal study. For further studies, see, e.g., [Takriti *et. al.* (2000); Sen *et. al.* (2006); Philpott and Schultz, (2006); Escudero *et. al.* (1996), (1998)] some of these already incorporate aspects of liberalized markets.

Within a liberalized market, power production and demand satisfaction do not necessarily need to be optimized jointly. Production capacity as well as demand can be submitted to an electricity pool market, e.g., to the spot market auction of a power exchange. For a producer it may be reasonable to consider some units (or even a single unit) and to optimize their (its) production schedule only with respect tot eh pool market; see, e.g., [Fleten and Kristoffersen, (2007b); Conejo *et. al.* (2004); Plazas *et. al.* (2005); Philpott and Schultz (2006)]. Also retailers and distributors can rely solely on the market to satisfy electricity demands; cf., e.g., [Fleten and Pettersen (2005)]. In either case, there is the question of optimal offer construction since electricity spot markets typically allow to submit offers which are sensitive to the effective market clearing prices; see, e.g., [Fleten and Pettersen (2005); Fleten and Kristoffersen (2007a); Philpott and Schultz (2006); Conejo *et. al.* (2002); Plazas *et. al.* (2005)].

However, spot market prices are known to be highly volatile, hence, the consideration of financial risk is indispensable in this case. Market price risk may be reduced by hedging instruments, i.e., by energy derivative products such as futures or options; cf., e.g., [Clewlow and Strickland (2000)]. For managing these hedging instruments, stochastic programming may again be an appropriate framework, in particular if an integrated handling of optimal production planning and risk management is adopted; see, e.g., [Fleten *et. al.* (2002); Hochreiter *et. al.* (2006)]. In the latter study it is shown that the integrated approach yields additional overall efficiency. Alternatively, bilateral delivery contracts between producers and distributors may be arranged to reduce the impact of spot market volatility to the respective revenues.

Finally, the trend towards renewable energy sources yields additional challenges for optimization in power. The consideration of physical aspects of electricity, production and transmission becomes more important; see, e.g., [Handschin *et. al.* (2005); Kuhn and Schultz (2008)] for a stochastic programming study on dispersed generation taking into account the topology of the transmission network.

(*ii*) *Other Applications*

Many real-world applications of mathematical programming could be reasonably extended to stochastic programming models since there are often some parameters that could be considered as uncertain. However, if the degree of uncertainty is low, the effort to pass from a deterministic to a stochastic model might not be worthwhile; the abandonment of other model assumptions and simplifications may be more rewarding. Furthermore, the availability of statistical information about the uncertainties is a necessary condition for a stochastic approach. And, moreover, the question arises whether it is then possible to solve a particular stochastic programming model, since the additional complexity induced by the stochastic is typically huge.

Notwithstanding these limitations stochastic programming has been successfully applied to numerous real-world problems. Important fields beside energy, where the stochastic programming approach has turned out to be essential or fruitful, are, e.g., finance [Ziemba (2003)], logistics [Powell and Topalogu (2003)], engineering, production, revenue management, airline planning, supply chain management, sports, catastrophe management, and others; see [Wallace and Ziemba (2005)] for a recent collection of case studies and reviews.

CHAPTER III

STOCHASTIC LINEAR PROGRAMMING PROBLEMS WITH CAUCHY AND EXTREME VALUE DISTRIBUTIONS

3.1 Introduction

There are many real-world applications where uncertainty is prevalent, such as finance, transportation, production planning, scheduling and other areas of management science. When dealing with real-world problems, the decision-maker often faces problems of optimizing several objectives at a time without knowing the values of some or all parameters. If these unknown parameters are considered as random variables, then the resulting problem can be treated as a stochastic single-objective or multi-objective programming problem. Stochastic programming models were first formulated by Dantzig (1955) who suggested a two-stage programming technique. This technique converts the stochastic problem into a deterministic problem and does not allow any constraints to be violated. Later, Charnes and Cooper (1959,1963) suggested a chance constrained programming technique which can be used to solve problems involving chance/probabilistic constraints. They suggested three models with different objective functions and probabilistic types of constraint:

- The E-model which maximizes the expected value of the objective function.
- The V-model which minimizes the generalized mean square of the objective function.
- The P-model which maximizes the probability of the aspiration level, i.e., the goal of the objective function.

Most applications of the probabilistic models assume a normal distributions have been considered for model coefficients [Goicoechea *et. al.* (1982); Infanger(1994)]. Recently, probabilistic models have been transformed into deterministic models in probabilistic programming by considering the stochastic parameters as exponential random variables [Biswal *et. al.* (1998)]. These deterministic models are either linear or non-linear depending on the problem and the stochastic parameters. In 1984, Stancu-Minasian (1984) discussed multi-objective linear programming problems involving random variables in coefficients of objective functions. Various books and research articles have been devoted to this field [Infanger (1994)]. Several methods for solving multi-objective stochastic linear programming problems have been proposed including the minimum-risk approach [Stancu-Minasian (1984)], the interactive method [Leclercq (1982)], the STRANGE method [Stancu-Minasian (1984)] and the Protrade method [Goicoechea *et. al.* (1982)]. The properties of stochastic programming problems and methods of obtaining optimal solutions have been described by various authors [Kall and Wallace (1994); Prekopa (1995); Kibzun and Kan (1996)]. A fuzzy approach to stochastic programming has been presented by Mohan and Nguyen (1997). Most recently an interactive fuzzy satisficing method has been developed by Sakawa *et. al.* (2003).

In this chapter we present some multi-objective stochastic linear programming models and their deterministic equivalents. The goal programming method [Ignizio (1982); Ignizio and Cavalier (1994)] has been used to solve the multi-objective linear programming model. This method has been widely applied in multi-objective decision making for the last 40 years. Its admirable history and promising future [Aouni and Kettani (2001)] have inspired many researchers to develop its theory and applications. The general structure of the achievement function of the goal programming [Romero (2004)], the mean-variance approach to stochastic goal programming [Ballestero (2001)] and decision-maker's preference modeling in stochastic goal programming [Aouni *et. al.* (2004)] are some of the useful models that have been developed in recent years. Fuzzy programming theory and applications have been reported by many researchers [Bellman and Zadeh (1970); Zimmermann (1978); Sakawa (1993)]. The relationship between goal programming and fuzzy programming has been presented by Mohamed (1997), and several other methods [Chankong and Haimes (1983); Stewart (1992)] have been used recently to solve multi-objective linear programming problems.

In the next two sections, we present the probabilistic multi-objective linear programming models.

3.2. Probabilistic model with Cauchy distribution

The mathematical model of a multi-objective probabilistic linear programming problem can be expressed as

$$Max: Z_k = \sum_{j=1}^{n} c_j^k x_j, \quad k = 1,2,\ldots,K \tag{3.2.1}$$

Subject to

$$\Pr\left(\sum_{j=1}^{n} a_{ij} x_j \le b_i\right) \ge 1 - \gamma_i, \; i = 1,2,\ldots,m \tag{3.2.2}$$

$$x_j \ge 0, \quad j = 1,2,\ldots,n \tag{3.2.3}$$

where $0 < \gamma_i < 1$ and is a given constant. It is assumed that the parameters a_{ij} and c_j are deterministic constants and only b_i are random variables having a Cauchy distribution. It is also given that the i^{th} random variable b_i has two known parameters α_i and β_i where the location parameter α_i is the median and β_i is the scale parameter of the random variable.

In the model the decision variables $x_j, j = 1,2,...n,$ are treated as deterministic variables. Let the probability density function of the random variable b_i be given by

$$f(b_i) = \frac{\beta_i}{\pi\left[\beta_i^2 + (b_i - \alpha_i)^2\right]}; \quad -\infty < b_i < \infty, \ \beta_i > 0. \tag{3.2.4}$$

Using the i^{th} constraint of the probabilistic problem, we restate the constraint (3.2.2) as

$$\Pr\left(b_i \geq \sum_{j=1}^{n} a_{ij} x_j\right) \geq 1 - \gamma_i, \ i = 1,2,...,m. \tag{3.2.5}$$

Let $y_i = \sum_{j=1}^{n} a_{ij} x_j$. Hence the probability constraint (3.2.5) can be further stated as

$$\int_{y_i}^{\infty} \frac{\beta_i}{\pi\left[\beta_i^2 + (b_i - \alpha_i)^2\right]} db_i \geq 1 - \gamma_i, \quad i = 1,2,...,m \tag{3.2.6}$$

which can be integrated as

$$\frac{1}{\pi}\left[\tan^{-1}\frac{(b_i - \alpha_i)}{\beta_i}\right]_{y_i}^{\infty} \geq 1 - \gamma_i, \ i = 1,2,...,m.$$

After substituting the limits of the integration, we find

$$\frac{\pi}{2}-\tan^{-1}\frac{(y_i-\alpha_i)}{\beta_i}\geq(1-\gamma_i)\pi,\ i=1,2,\ldots,m.$$

which can be further simplified to

$$-\tan^{-1}\frac{(y_i-\alpha_i)}{\beta_i}\geq(\pi/2-\pi\,\gamma_i),\ i=1,2,\ldots,m.$$

Taking the tangent of both sides, we find

$$\frac{(y_i-\alpha_i)}{\beta_i}\geq\tan(\pi\,\gamma_i-\pi/2),\quad i=1,2,\ldots,m.$$

or $$y_i\leq\alpha_i+\beta_i\tan(\pi\gamma_i-\pi/2),\ i=1,2,\ldots,m. \tag{3.2.7}$$

Finally, this can be expressed as a linear constraint in the form

$$\sum_{j=1}^{n}a_{ij}x_j\leq\alpha_i+\beta_i\tan(\pi\,\gamma_i-\pi/2),\ i=1,2,\ldots,m. \tag{3.2.8}$$

Hence the deterministic multi-objective linear programming model can be expressed as

$$
\left.
\begin{aligned}
&Max: Z_k = \sum_{j=1}^{n} c_j^k x_j, \quad k = 1,2,\ldots,K \\
&\text{Subject to} \\
&\qquad \sum_{j=1}^{n} a_{ij} x_j \le \alpha_i + \beta_i \tan(\pi \gamma_i - \pi/2),\ i = 1,2,\ldots,m. \\
&\qquad\qquad x_j \ge 0,\ j = 1,2,\ldots,n.
\end{aligned}
\right\} \quad (3.2.9)
$$

This deterministic multi-objective linear programming model can be solved using the fuzzy programming or the goal programming method.

3.3 Probabilistic linear programming problem with join constraint for Cauchy distribution

The mathematical model of a multi-objective probabilistic linear programming problem with a joint constraint can be expressed as

$$Max: Z_k = \sum_{j=1}^{n} c_j^k x_j, \quad k = 1,2,\ldots,K \qquad (3.3.1)$$

Subject to

$$\Pr\left(\sum_{j=1}^{n} a_{1j}x_j \le b_1, \sum_{j=1}^{n} a_{2j}x_j \le b_2, \ldots, \sum_{j=1}^{n} a_{mj}x_j \le b_m\right) \ge 1-\gamma$$

(3.3.2)

$$x_j \ge 0,\ j = 1,2,\ldots,n. \qquad (3.3.3)$$

where $0<\gamma<1$ and is known with certainty. It is assumed that $b_i, i=1,2,...m,$ are independent Cauchy random variables with known distribution. It is also given that the i^{th} random variable b_i has two known parameters α_i and β_i where the location parameter α_i is the median and β_i is the scale parameter of the random variable.

Let

$$y_i = \sum_{j=1}^{n} a_{ij}x_j,\ i = 1,2,\ldots,m \qquad (3.3.4)$$

Now, the joint probability constraint (3.3.2) can be written

$$\Pr(b_1 \ge y_1, b_2 \ge y_2, \ldots, b_m \ge y_m) \ge 1-\gamma. \qquad (3.3.5)$$

Since $b_i, i=1,2,\ldots,m$, are independent random variables, the above joint constraint can be expressed as

$$\prod_{i=1}^{m} \Pr(b_i \ge y_i) \ge 1-\gamma \qquad (3.3.6)$$

where

$$\Pr(b_i \geq y_i) = \int_{y_i}^{\infty} \frac{\beta_i}{\pi\left[\beta_i^2 + (b_i - \alpha_i)^2\right]} db_i. \tag{3.3.7}$$

After integration, this becomes

$$\Pr(b_i \geq y_i) = \frac{1}{\pi}\left[\tan^{-1}\frac{(b_i - \alpha_i)}{\beta_i}\right]_{y_i}^{\infty}$$

Taking the limits on the integration we find

$$\Pr(b_i \geq y_i) = \frac{1}{\pi}\left[\frac{\pi}{2} - \tan^{-1}\frac{(y_i - \alpha_i)}{\beta_i}\right].$$

Hence the joint probabilistic constraint can be transformed into a deterministic constraint:

$$\prod_{i=1}^{m}\frac{1}{\pi}\left[\frac{\pi}{2} - \tan^{-1}\frac{(y_i - \alpha_i)}{\beta_i}\right] \geq 1 - \gamma$$

or
$$\prod_{i=1}^{m}\left[\frac{\pi}{2} - \tan^{-1}\frac{(y_i - \alpha_i)}{\beta_i}\right] \geq (1 - \gamma)\pi^m.$$

which can be further simplified to

$$\prod_{i=1}^{m}\left(\frac{\pi}{2} - t_i\right) \geq (1 - \gamma)\pi^m \tag{3.3.8}$$

where $\tan(t_i) = \dfrac{(y_i - \alpha_i)}{\beta_i}.$

Hence the deterministic multi-objective model can be expressed as

$$\left.\begin{aligned}
&Max: Z_k = \sum_{j=1}^{n} c_j^k x_j, \quad k = 1,2,\ldots,K \\
&\text{Subject to} \\
&\sum_{j=1}^{n} a_{ij} x_j = \alpha_i + \beta_i \tan(t_i), \;\; i = 1,2,\ldots,m. \\
&\prod_{i=1}^{m} \left(\frac{\pi}{2} - t_i\right) \geq (1-\gamma)\pi^m \\
&x_j \geq 0, \; j = 1,2,\ldots,n.
\end{aligned}\right\} \qquad (3.3.9)$$

This deterministic multi-objective non-linear programming model can be solved using the fuzzy programming method.

3.4 Probabilistic model with extreme value distribution

The mathematical model of a probabilistic linear programming problem can be expressed as

$$Max: Z_k = \sum_{j=1}^{n} c_j^k x_j, \quad k = 1,2,\ldots,K \qquad (3.4.1)$$

Subject to

$$\Pr\left(\sum_{j=1}^{n} a_{ij}x_j \le b_i\right) \ge 1-\gamma_i,\ i=1,2,\ldots,m \tag{3.4.2}$$

$$x_j \ge 0,\ j=1,2,\ldots,n \tag{3.4.5}$$

where $0<\gamma_i<1$ and is a given constant. It is assumed that the parameters a_{ij} and c_j are deterministic constants and only b_i are random variables with an extreme value distribution. It is also given that the i^{th} random variable b_i has two known parameters α_i and β_i where the location parameter α_i is the mode and β_i is the scale parameter of the random variable. In the model the decision variables x_j are treated as deterministic variable. Let the probability density function of b_i be

$$f(b_i)=\frac{1}{\beta_i}e^{-(b_i-\alpha_i)/\beta_i}\ e^{-e^{-(b_i-\alpha_i)/\beta_i}},\quad -\infty<b_i<\infty,\ \beta_i>0. \tag{3.4.4}$$

We restate the i^{th} constraint (3.4.2) as

$$\Pr\left(b_i \ge \sum_{j=1}^{n} a_{ij}x_j\right) \ge 1-\gamma_i,\ i=1,2,\ldots,m \tag{3.4.5}$$

Let $\qquad y_i = \sum_{j=1}^{n} a_{ij}x_j.$

Hence the probability constraint can be expressed as

$$\int_{y_i}^{\infty} \frac{1}{\beta_i} e^{-(b_i-\alpha_i)/\beta_i} e^{-e^{-(b_i-\alpha_i)/\beta_i}} db_i \geq 1-\gamma_i, \; i=1,2,\ldots,m. \tag{3.4.6}$$

Let $\quad e^{-(b_i-\alpha_i)/\beta_i} = t_i$.

Then the integral (3.4.6) can be written as

$$\int_0^{e^{-(y_i-\alpha_i)/\beta_i}} e^{-t_i}\, dt_i \geq 1-\gamma_i, \; i=1,2,\ldots,m \tag{3.4.7}$$

Taking the limits on the integration, this can be simplified to

$$\left(-e^{-t_i}\right)_0^{e^{-(y_i-\alpha_i)/\beta_i}} \geq 1-\gamma_i, \; i=1,2,\ldots,m$$

or $$1-e^{-e^{-(y_i-\alpha_i)/\beta_i}} \geq (1-\gamma_i), \; i=1,2,\ldots,m$$

which can be rearranged as follows:

$$e^{-e^{-(y_i-\alpha_i)/\beta_i}} \leq \gamma_i, \; i=1,2,\ldots,m$$

By taking logarithms on both sides this becomes

$$-e^{-(y_i-\alpha_i)/\beta_i} \leq \log(\gamma_i), \; i=1,2,\ldots,m$$

or $$e^{-(y_i-\alpha_i)/\beta_i} \geq -\log(\gamma_i), \; i=1,2,\ldots,m \tag{3.4.8}$$

Taking further logarithms we obtain

$$-\frac{(y_i-\alpha_i)}{\beta_i} \geq \log[-\log(\gamma_i)], i=1,2,\ldots,m$$

or
$$\frac{(y_i-\alpha_i)}{\beta_i} \leq -\log[-\log(\gamma_i)], i=1,2,\ldots,m$$

which can be rearranged as follows:

$$y_i \leq \alpha_i - \beta_i \log[-\log(\gamma_i)],\ i=1,2,\ldots,m$$

Finally, this can be expressed as linear constraints:

$$\sum_{j=1}^{n} a_{ij}x_j \leq \alpha_i - \beta_i \log[-\log(\gamma_i)],\ i=1,2,\ldots,m. \tag{3.4.9}$$

Hence the deterministic multi-objective linear programming model is

$$\left.\begin{aligned} &Max: Z_k = \sum_{j=1}^{n} c_j^k x_j, \quad k=1,2,\ldots,K \\ &\text{Subject to} \\ &\sum_{j=1}^{n} a_{ij}x_j \leq \alpha_i - \beta_i \log[-\log(\gamma_i)],\ i=1,2,\ldots,m. \\ &x_j \geq 0,\ j=1,2,\ldots,n \end{aligned}\right\} \tag{3.4.10}$$

This deterministic multi-objective linear programming model can be solved using the fuzzy programming or the goal programming method.

3.5 Probabilistic linear programming problem with joint constraints for extreme value distribution

The mathematical model of a multi-objective probabilistic linear programming problem with a joint constraint [Aouni *et. al.* (2004)] can be expressed as

$$Max: Z_k = \sum_{j=1}^{n} c_j^k x_j, \quad k = 1,2,..., K \tag{3.5.1}$$

Subject to

$$\Pr\left(\sum_{j=1}^{n} a_{1j} x_j \le b_1, \sum_{j=1}^{n} a_{2j} x_j \le b_2, ..., \sum_{j=1}^{n} a_{mj} x_j \le b_m \right) \ge 1 - \gamma \tag{3.5.2}$$

$$x_j \ge 0, \ j = 1,2,...,n. \tag{3.5.3}$$

where $0 < \gamma < 1$, and is known with certainty. It is assumed that $b_i, i = 1,2,...,m$, are independent extreme value random variables with known distribution. It is also given that the i^{th} random variable b_i has two known parameters α_i and β_i where the location parameter α_i is the mode and β_i is the scale parameter of the random variable.

Let $$y_i = \sum_{j=1}^{n} a_{ij} x_j, \ i = 1,2,....,m \tag{3.5.4}$$

Now, the joint probability constraint (3.5.2) can be written

$$\Pr(b_1 \ge y_1, b_2 \ge y_2, \ldots, b_m \ge y_m) \ge 1-\gamma. \qquad (3.5.5)$$

Since $b_i, i = 1,2,\ldots,m$, are independent random variables, the above joint constraint can be expressed as

$$\prod_{i=1}^{m} \Pr(b_i \ge y_i) \ge 1-\gamma \qquad (3.5.6)$$

where $\Pr(b_i \ge y_i) = \int_{y_i}^{\infty} \frac{1}{\beta_i} e^{-(b_i-\alpha_i)/\beta_i}\; e^{-e^{-(b_i-\alpha_i)/\beta_i}}\, db_i$.

$$(3.5.7)$$

This can be written as

$$\prod_{i=1}^{m}\left[\int_{y_i}^{\infty} \frac{1}{\beta_i} e^{-(b_i-\alpha_i)/\beta_i}\; e^{-e^{-(b_i-\alpha_i)/\beta_i}}\, db_i\right] \ge 1-\gamma$$

which can be put in the form

$$\prod_{i=1}^{m}\left[\int_{0}^{e^{-(y_i-\alpha_i)/\beta_i}} e^{-t_i}\, dt_i\right] \ge 1-\gamma$$

where $t_i = e^{-(y_i-\alpha_i)/\beta_i}$.

After integration, this can be simplified to

$$\prod_{i=1}^{m}\left[1 - e^{-e^{-(y_i-\alpha_i)/\beta_i}}\right] \ge 1-\gamma$$

or $$\prod_{i=1}^{m}\left[1-e^{s_i}\right]\geq 1-\gamma \tag{3.5.8}$$

where $s_i = e^{-(y_i-\alpha_i)/\beta_i}$.

Hence we can write

$$\sum_{j=1}^{n} a_{ij}x_j = \alpha_i - \beta_i \log(s_i),\ i=1,2,\ldots,m \tag{3.5.9}$$

Thus the deterministic model can be expressed as

$$\left.\begin{array}{l}
Max: Z_k = \sum_{j=1}^{n} c_j^k x_j, \quad k=1,2,\ldots,K \\[2ex]
\text{Subject to} \\[2ex]
\prod_{i=1}^{m}\left[1-e^{s_i}\right]\geq 1-\gamma \\[2ex]
\text{where} \\[2ex]
\sum_{j=1}^{n} a_{ij}x_j = \alpha_i - \beta_i \log(s_i),\ i=1,2,\ldots,m \\[2ex]
x_j \geq 0,\ j=1,2,\ldots,n.
\end{array}\right\} \tag{3.5.10}$$

This deterministic multi-objective non-linear programming model can be solved using the fuzzy programming method.

3.6 Numerical example 1

A multi-objective linear programming problem is presented to illustrate the solution procedure. The problem is as follows.

A firm produces four electronic products A, B, C and D. Product A has a net return of $20 per unit, product B returns $18 per unit, product C returns $16 per unit and product D returns $14 per unit. Product A requires 5 h per unit assembly time, B requires 4 h per unit assembly time, C requires 3h per unit assembly time and D requires 2 h per unit assembly time. The total assembly time available is around 300 h per week but some overtime is possible. Further, the assembly time available per week is a Cauchy random variable with two known parameters α=300 and β=2.0. However, if overtime is utilized, the net return on all the products is reduced by $q per unit produced on overtime. Under the present contract the firm must supply the customer with a minimum of 30 units per week on all products. The following decisions have been made by the manager of the firm: about 300 h of regular time is available per week; weekly overtime (OT) is minimized at about 100 h; weekly profit (P) is maximized at about $2500.

The model is formulated as follows:

x_1 = number of units of product A produced per week in regular time.

x_2 = number of units of product A produced per week in over time.

x_3 = number of units of product B produced per week in regular time.

x_4 = number of units of product B produced per week in over time.

x_5 = number of units of product C produced per week in regular time.

x_6 = number of units of product C produced per week in over time.

x_7 = number of units of product D produced per week in regular time.

x_8 = number of units of product D produced per week in over time.

Then the multi-objective linear programming model is

$$Min: OT = 5x_2 + 4x_4 + 3x_6 + 2x_8 \tag{3.6.1}$$

$$Max: P = 20x_1 + 19x_2 + 18x_3 + 17x_4 + 16x_5 + 15x_6 + 14x_7 + 13x_8 \tag{3.6.2}$$

Subject to

$$\Pr(5x_1 + 4x_3 + 3x_5 + 2x_7 \le b) \ge 0.90 \tag{3.6.3}$$

$$x_1 + x_2 \ge 30 \tag{3.6.4}$$

$$x_3 + x_4 \ge 30 \tag{3.6.5}$$

$$x_5 + x_6 \ge 30 \tag{3.6.6}$$

$$x_7 + x_8 \ge 30 \tag{3.6.7}$$

$$x_j \ge 0, \quad j = 1,2,\ldots,8. \tag{3.6.8}$$

The random variable b has a Cauchy distribution with two known parameters α =300 and β = 2.

From equation (3.2.9) the deterministic model of the probabilistic problem is

$$Min: OT = 5x_2 + 4x_4 + 3x_6 + 2x_8 \tag{3.6.9}$$

$$Max: P = 20x_1 + 19x_2 + 18x_3 + 17x_4 + 16x_5 + 15x_6 + 14x_7 + 13x_8 \tag{3.6.10}$$

Subject to

$$5x_1 + 4x_3 + 3x_5 + 2x_7 \le 294 \tag{3.6.11}$$

$$x_1 + x_2 \ge 30 \tag{3.6.12}$$

$$x_3 + x_4 \ge 30 \tag{3.6.13}$$

$$x_5 + x_6 \geq 30 \tag{3.6.14}$$

$$x_7 + x_8 \geq 30 \tag{3.6.15}$$

$$x_j \geq 0, \quad j = 1,2,\ldots,8. \tag{3.6.16}$$

Then the deterministic multi-objective linear programming problem is formulated as a goal programming model and solved using a multi-phase simplex algorithm. The goal programming model is as follows.

Find $x_1, x_{2,\ldots,} x_8$ so as to

Lexicographically

$$Min: a = \{(p_1 + n_2 + n_3 + n_4 + n_5), p_6, n_7\} \tag{3.6.17}$$

Subject to

$$5x_1 + 4x_3 + 3x_5 + 2x_7 + n_1 - p_1 = 294 \tag{3.6.18}$$

$$x_1 + x_2 + n_2 - p_2 = 30 \tag{3.6.19}$$

$$x_3 + x_4 + n_3 - p_3 = 30 \tag{3.6.20}$$

$$x_5 + x_6 + n_4 - p_4 = 30 \tag{3.6.21}$$

$$x_7 + x_8 + n_5 - p_5 = 30 \tag{3.6.22}$$

$$5x_2 + 4x_4 + 3x_6 + 2x_8 + n_6 - p_6 = 100 \tag{3.6.23}$$

$$20x_1 + 19x_2 + 18x_3 + 17x_4 + 16x_5 + 15x_6 + 14x_7 + 13x_8 + n_7 - p_7 = 2500 \quad (3.6.24)$$

$$n_i, p_i = 0,1,2,..., \; i = 1,2,...,7, \; x_j = 0,1,2,..., \; j = 1,2,...,8 \quad (3.6.25)$$

The linear integer goal programming problem has two alternative optimal solutions: $X^{(1)} = (8,22,26,4,30,0,30,0)$ and $X^{(2)} = (6,24,30,0,28,2,30,0)$ where the achievement vector a = (0, 26,486), (table 1 and 2).

3.7 Numerical example 2

The parameters of example 2 are the same as those of example 1 except that the assembly time available per week, i.e., b, is an extreme value random variable with two known parameters α = 300 and β = 2.0. With this assumption, the deterministic model of the probabilistic problem is obtained using equation (50) as

$$Min: OT = 5x_2 + 4x_4 + 3x_6 + 2x_8 \quad (3.7.1)$$

$$Max: P = 20x_1 + 19x_2 + 18x_3 + 17x_4 + 16x_5 + 15x_6 + 14x_7 + 13x_8 \quad (3.7.2)$$

Subject to

$$5x_1 + 4x_3 + 3x_5 + 2x_7 \leq 298 \quad (3.7.3)$$

$$x_1 + x_2 \geq 30 \quad (3.7.4)$$

$$x_3 + x_4 \geq 30 \tag{3.7.5}$$

$$x_5 + x_6 \geq 30 \tag{3.7.6}$$

$$x_7 + x_8 \geq 30 \tag{3.7.7}$$

$$x_j \geq 0, \quad j = 1,2,\ldots,8. \tag{3.7.8}$$

Then the deterministic multi-objective linear programming problem is formulated as a goal programming model and solved using a multi-phase simplex algorithm. The goal programming model is as follows.

Find $x_1, x_{2,\ldots,} x_8$ so as to

Lexicographically

$$Min: a = \{(p_1 + n_2 + n_3 + n_4 + n_5), p_6, n_7\} \tag{3.7.9}$$

Subject to

$$5x_1 + 4x_3 + 3x_5 + 2x_7 + n_1 - p_1 = 298 \tag{3.7.10}$$

$$x_1 + x_2 + n_2 - p_2 = 30 \tag{3.7.11}$$

$$x_3 + x_4 + n_3 - p_3 = 30 \tag{3.7.12}$$

$$x_5 + x_6 + n_4 - p_4 = 30 \tag{3.7.13}$$

$$x_7 + x_8 + n_5 - p_5 = 30 \tag{3.7.14}$$

$$5x_2 + 4x_4 + 3x_6 + 2x_8 + n_6 - p_6 = 100 \tag{3.7.15}$$

$$20x_1 + 19x_2 + 18x_3 + 17x_4 + 16x_5 + 15x_6 + 14x_7 + 13x_8 + n_7 - p_7 = 2500 \quad (3.7.16)$$

$$n_i, p_i = 0,1,2,.... \; i = 1,2,...,7, \quad x_j = 0,1,2,.... \quad j = 1,2,...,8 \quad (3.7.17)$$

The linear integer goal programming problem has two alternative optimal solutions: $X^{(1)} = (6,24,30,0,30,0,29,1)$and $X^{(2)} = (8,22,27,3,30,0,30,0)$where the achievement vector a = (0, 22, 485) (tables 3 and 4).

Table 1. Solution 1 of integer goal programming problem 1.

No.	Achievement Vector	Solution Vector	Positive Vector	Negative Vector
1	0.0000	8.000	0.0000	0.0000
2	26.0000	22.0000	0.0000	0.0000
3	486.0000	26.0000	0.0000	0.0000
4		4.0000	0.0000	0.0000
5		30.0000	0.0000	0.0000
6		0.0000	26.0000	0.0000
7		30.0000	0.0000	486.0000
8		0.0000	0.0000	0.0000

Table 2. Solution 2 of integer goal programming problem 1.

No.	Achievement Vector	Solution Vector	Positive Vector	Negative Vector
1	0.0000	6.000	0.0000	0.0000
2	22.0000	24.0000	0.0000	0.0000
3	485.0000	30.0000	0.0000	0.0000
4		0.0000	0.0000	0.0000
5		30.0000	0.0000	0.0000
6		0.0000	22.0000	0.0000
7		29.0000	0.0000	485.0000
8		1.0000	0.0000	0.0000

Table 3. Solution 1 of integer goal programming problem 2.

No.	Achievement Vector	Solution Vector	Positive Vector	Negative Vector
1	0.0000	6.000	0.0000	0.0000
2	26.0000	24.0000	0.0000	0.0000
3	486.0000	30.0000	0.0000	0.0000
4		0.0000	0.0000	0.0000
5		28.0000	0.0000	0.0000
6		2.0000	26.0000	0.0000
7		30.0000	0.0000	486.0000
8		0.0000	0.0000	0.0000

Table 4. Solution 2 of integer goal programming problem 2.

No.	Achievement Vector	Solution Vector	Positive Vector	Negative Vector
1	0.0000	8.000	0.0000	0.0000
2	22.0000	22.0000	0.0000	0.0000
3	485.0000	27.0000	0.0000	0.0000
4		3.0000	0.0000	0.0000
5		30.0000	0.0000	0.0000
6		0.0000	22.0000	0.0000
7		30.0000	0.0000	485.0000
8		0.0000	0.0000	0.0000

CHAPTER IV

PROBABILISTIC LINEAR PROGRAMMING PROBLEMS INVOLVING NORMAL AND LOG-NORMAL RANDOM VARIABLES WITH A JOINT CONSTRAINT

4.1 Introduction

Charnes and Cooper (1959, 1963) first introduced the chance-constrained programming model which is known as probabilistic programming. They suggested three models with different types of objective functions and probabilistic constraints that maximize the expected value of the objective function (the E-model), minimize the generalized mean square of the objective function (the V-model) or maximize the probability that the aspiration level reaches the goal of the objective function (the P-model). Most of the applications of the probabilistic models assume a normal distribution for the model coefficients [Kambo (1984)]. However, other distributions have been considered for the model coefficients by a number of researchers [Stancu-Minasian and Wets (1976); Vajda (1972); Kall and Wallace (1994)]. Goicoehea *et. al.* (1982) described probabilistic models involving uniform, exponential, normal and other random variables. Since the introduction of chance-constrained programming, various models have been suggested by several researchers, and a bibliography has been compiled by Stancu Minasian and Wets (1976).

It is assumed that some or all of the a_{ij}, b_i and c_j in a multi-objective probabilistic linear programming problem may be random variables rather than fixed constants. Each c_j is a per unit net profit determined from net revenues or cost for the j^{th} output x_j. However, the production cost may be unknown and the net profit on each product may be unknown and the net profit on each product may be a random variable with an estimated mean and variance. Furthermore, because of quality control, time restrictions and other unpredictable aspects, the exact required inputs a_{ij} per unit of the output may not be known with certainty. Again, the assumed constants may be random variables with estimated means and variances. It is also impossible for the decision-maker to know exactly how many man-hours or resource unit b_i will be available during any production period because of employee absenteeism machine breakdown, load shading etc. Thus each b_i can be assumed to be a random variable whose mean and variance can be estimated. In a probabilistic linear programming problem, the probability of satisfying a constraint is known with certainty. There may be several probabilistic constraints in such a problem. It may not be so easy for a decision-maker to know the probabilities of the satisfaction levels of all the probabilistic constraints.

However, the decision-maker can easily state the probability of the satisfaction level of the joint constraints from his or her past experience. Miller and Wagner (1965) and Jagannathan (1974) have presented a single-objective probabilistic model in which the model parameters are assumed to be normal random variables. Because of the nature of the situation, a production planning problem [Johnson and Montgomory (1974); Lai and Hwang (1992, 1994)] for a captive overhauling plant can be considered as a multi-objective probabilistic linear programming problem.

4.2 Single-objective probabilistic linear programming problem with a joint constraint

The mathematical model for a single-objective probabilistic linear programming problem with a joint constraint can be presented as follows:

$$Max: Z = \sum_{j=1}^{n} c_j x_j \tag{4.2.1}$$

Subject to

$$\Pr\left(\sum_{j=1}^{n} a_{1j}x_j \le b_1, \sum_{j=1}^{n} a_{2j}x_j \le b_2, \ldots, \sum_{j=1}^{n} a_{mj}x_j \le b_m\right) \ge 1-\alpha \tag{4.2.2}$$

$$x_j \ge 0,\ j = 1,2,\ldots,n \tag{4.2.3}$$

where and is known with certainty. It is assumed that $a_{ij},\ i = 1,2,..,m,\ j = 1,2,...,n,\ \ b_i,\ i = 1,2,....,m,\ and\ c_j,\ j = 1,2,...,n$ are specific random variables.

4.3 Multi-objective probabilistic linear programming problem with a joint constraint

The mathematical model for a multi-objective probabilistic linear programming problem with a joint constraint can be presented as follows:

$$Max : Z_k = \sum_{j=1}^{n} c_j^k x_j,\ k = 1,2,...,k \tag{4.3.1}$$

Subject to

$$\Pr\left(\sum_{j=1}^{n} a_{1j} x_j \le b_1, \sum_{j=1}^{n} a_{2j} x_j \le b_2, ..., \sum_{j=1}^{n} a_{mj} x_j \le b_m \right) \ge 1 - \alpha \tag{4.3.2}$$

$$x_j \ge 0,\ j = 1,2,...,n \tag{4.3.3}$$

where $0 < \alpha < 1$ and is known with certainty. It is assumed that $a_{ij},\ i = 1,2,..,m,\ j = 1,2,...,n;\ b_i = 1,2,...,m$ and $c_j^k,\ j = 1,2,...,n$ $k = 1,2,...,K$ are specific random variables.

4.4 Random Parameters with Joint Constraint

(i) When b_i are normal random variables

It is assumed that b_i, $i = 1,2,\ldots,m$ are independent normal random variables with $E(b_i) = \mu_i$, $Var(b_i) = \sigma_i^2$, $i = 1,2,\ldots,m$ and c_j^k, $j = 1,2,\ldots,n$, are random variables with known means for all values of k. For simplicity, $a_{ij}, i = 1,2,\ldots m; j = 1,2,\ldots n$ are assumed to be deterministic constants. The probability density function pdf of the i^{th} random variable $b_i, i = 1,2,\ldots m$, is given by

$$f_i(b_i) = \frac{1}{\sqrt{2\pi}\sigma_i}\exp\left[-\frac{1}{2}\left(\frac{b_i - \mu_i}{\sigma_i}\right)^2\right], -\infty < b_i < \infty, \sigma_i > 0$$

(4.3.4)

Let

$$y_i = \sum_{j=1}^{n} a_{ij} x_j, \; i = 1,2,\ldots,m \qquad (4.3.5)$$

where $y_i \geq 0$. Now, the joint probability constraints (4.3.2) can be written as

$$\Pr(b_1 \geq y_1, b_2 \geq y_2, \ldots, b_m \geq y_m) \geq 1 - \alpha.$$

(4.3.6)

Since $b_i, i = 1,2,...m$ are independent random variables, the above joint constraints can be expressed as

$$\prod_{i=1}^{m} \Pr(b_i \geq y_i) \geq 1 - \alpha. \tag{4.3.7}$$

where $\Pr(b_i \geq y_i) = \Pr\left(\frac{b_i - \mu_i}{\sigma_i} \geq \frac{y_i - \mu_i}{\sigma_i}\right)$, $i = 1,2,...,m.$

whichhis can be simplified to

$$\Pr(b_i \geq y_i) = 1 - \Phi\left(\frac{y_i - \mu_i}{\sigma_i}\right), \ i = 1,2,...,m.$$

Hence the joint constraints can be simplified to

$$\prod_{i=1}^{m} \left[1 - \Phi\left(\frac{y_i - \mu_i}{\sigma_i}\right)\right] \geq 1 - \alpha. \tag{4.3.8}$$

The equivalent multi-objective deterministic model of the probabilistic problem (4.3.1)-(4.3.3) can be stated as the E-model given below:

$$
\left.
\begin{aligned}
& Max: Z_k = \sum_{j=1}^{n} E\left[c_j^k\right] x_j,\ k = 1,2,\dots,K \\
& \text{Subject to} \\
& \prod_{i=1}^{m}\left[1 - \Phi\left(\frac{y_i - \mu_i}{\sigma_i}\right)\right] \geq 1 - \alpha. \\
& \sum_{j=1}^{n} a_{ij} x_j = y_i,\ i = 1,2,\dots,m \\
& y_i \geq 0,\ i = 1,2,\dots,m \\
& x_j \geq 0,\ j = 1,2,\dots,m
\end{aligned}
\right\} \qquad (4.3.9)
$$

If $k=1$, the problem is treated as a single-objective mathematical programming problem which can be solved using standard mathematical programming techniques.

(ii) When a_{ij} are normal random variables

It is assumed that $a_{ij}, i = 1,2,\dots m; j = 1,2,\dots n$, are independent normal random variables, where

$$E(a_{ij}) = \mu_{ij},\ Var(a_{ij}) = \sigma_{ij}^2,\ i = 1,2,\dots,m,\ j = 1,2,\dots,n$$

where $b_i, i = 1,2,\dots m$, are deterministic constants and $c_j^k,\ j = 1,2,\dots,n,$ are random variables with known means for all values of k.

Let

$$y_i = \sum_{j=1}^{n} a_{ij} x_j, \; i = 1,2,...,m$$

where $y_i \geq 0, i = 1,2,...,m$. It is given that

$$E(y_i) = M_i, \quad Var(y_i) = S_i^2, \; i = 1,2,...,m.$$

The pdf of the i^{th} random variable $y_i, i = 1,2,...,m,$ is given by

$$f_i(y_i) = \frac{1}{\sqrt{2\pi} S_i} \exp\left[-\frac{1}{2}\left(\frac{y_i - M_i}{S_i} \right)^2 \right], \; -\infty < y_i < \infty \tag{4.3.10}$$

where $S_i > 0$.

Now, the joint probability constraints (4.3.2) can be written as

$$\Pr(y_1 \leq b_1, y_2 \leq b_2, ..., y_m \leq b_m) \geq 1 - \alpha. \tag{4.3.11}$$

Since the y_i, $i = 1,2,...,m$, are independent normal random variable, the above joint constraints can be expressed as

$$\prod_{i=1}^{m} \Pr(y_i \leq b_i) \geq 1 - \alpha$$

where

$$\Pr(y_i \le b_i) = \Pr\left(\frac{y_i - M_i}{S_i} \le \frac{b_i - M_i}{S_i}\right), \; i = 1,2,\ldots,m.$$

which can be simplified to

$$\Pr(y_i \le b_i) = \Phi\left(\frac{b_i - M_i}{S_i}\right), \; i = 1,2,\ldots,m.$$

It is assumed that M_i is a positive number. Therefore we can write

$$\Pr(y_i \le b_i) = \Phi\left(\frac{b_i - M_i}{S_i}\right), \; i = 1,2,\ldots,m.$$

Hence the joint constraints can be simplified to

$$\prod_{i=1}^{m} \Phi\left(\frac{b_i - M_i}{S_i}\right) \ge 1 - \alpha. \tag{4.3.12}$$

Let $\quad t_i = \Phi\left(\dfrac{b_i - M_i}{S_i}\right), \; i = 1,2,\ldots,m$

where $t_i \ge 0, i = 1,2,\ldots,m.$ The equivalent multi-objective deterministic model of the probabilistic problem (4.3.1)-(4.3.3) can be stated as the E-model given below:

$$\left.\begin{aligned}
&Max: Z_k = \sum_{j=1}^{n} E\left[c_j^k\right]x_j,\ k = 1,2,\ldots,K \\
&\text{Subject to} \\
&\prod_{i=1}^{m} t_i \geq 1-\alpha \\
\text{where}\quad & t_i = \Phi\left(\frac{b_i - M_i}{S_i}\right),\ i = 1,2,\ldots,m \\
&t_i \geq 0,\ i = 1,2,\ldots,m. \\
&x_j \geq 0,\ j = 1,2,\ldots,n.
\end{aligned}\right\} \qquad (4.3.13)$$

(iii) When The a_{ij} and b_i are normal random variable

It is assumed that $a_{ij}, i = 1,2,\ldots,m,\ j = 1,2,\ldots,n,$ and $b_i, i = 1,2,\ldots,m,$ are independent normal random variables where

$$E(b_i) = \mu_i,\ Var(b_i) = \sigma_i^2,\ i = 1,2,\ldots,m$$

$$E(a_{ij}) = \mu_{ij},\ Var(a_{ij}) = \sigma_{ij}^2,\ i = 1,2,\ldots,m,\ j = 1,2\ldots,n$$

and $c_j^k, j = 1,2,\ldots,n,$ are random variables with known means for all values of k.

Let

$$y_i = \sum_{j=1}^{n} a_{ij}x_j - b_i, \; i = 1,2,...,m \tag{4.3.14}$$

where $\quad y_i \geq 0, \; i = 1,2,...,m.$

It is given that $E(y_i) = M_i, \; Var(y_i) = S_i^2, \; i = 1,2,...,m.$

The pdf of the i^{th} random variable $y_i, \; i = 1,2,...,m,$ is given by

$$f_i(y_i) = \frac{1}{\sqrt{2\pi}S_i}\exp\left[-\frac{1}{2}\left(\frac{y_i - M_i}{S_i}\right)^2\right], \; -\infty < y_i < +\infty$$

where $\quad S_i > 0.$

Now the joint probability constraints (4.3.2) can be written as

$$\Pr(y_1 \leq 0, y_2 \leq 0,..., y_m \leq 0) \geq 1 - \alpha. \tag{4.3.15}$$

Since $y_i, \; i = 1,2,...,m,$ are independent normal random variables, these joint constraints can be expressed as

$$\prod_{i=1}^{m} \Pr(y_i \leq 0) \geq 1 - \alpha$$

where

$$\Pr(y_i \leq 0) = \Pr\left(\frac{y_i - M_i}{S_i} \leq -\frac{M_i}{S_i}\right), \; i = 1,2,...,m.$$

which can be simplified to

$$\Pr(y_i \le 0) = \Phi\left(\frac{-M_i}{S_i}\right), \quad i = 1,2,\ldots,m.$$

It is assumed that M_i is a positive number. Therefore we can write

$$\Pr(y_i \le 0) = 1 - \Phi\left(\frac{M_i}{S_i}\right), \quad i = 1,2,\ldots,m.$$

Hence the joint constraints can be simplified to

$$\prod_{i=1}^{m}\left[1 - \Phi\left(\frac{M_i}{S_i}\right)\right] \ge 1 - \alpha. \tag{4.3.16}$$

Let

$$t_i = 1 - \Phi\left(\frac{M_i}{S_i}\right), \quad i = 1,2,\ldots,m$$

where $t_i \ge 0, i = 1,2,\ldots,m.$ The equivalent multi-objective deterministic model of the probabilistic problem (4.3.1)-(4.3.3) can be stated as the E-model below:

$$
\left.
\begin{aligned}
& Max: Z_k = \sum_{j=1}^{n} E\left[c_j^k\right] x_j, \ k = 1,2,\ldots,K \\
& \text{Subject to} \\
& \prod_{i=1}^{m} t_i \geq 1-\alpha \\
& \text{where} \quad t_i = 1-\Phi\left(\frac{M_i}{S_i}\right), i = 1,2,\ldots,m \\
& t_i \geq 0, \ i = 1,2,\ldots,m. \\
& x_j \geq 0, \ j = 1,2,\ldots,n.
\end{aligned}
\right\} \quad (4.3.17)
$$

(*iv*) *When b_i are log-normal random variables*

It is assumed that the $b_i, i = 1,2,\ldots,m$ are independent log-normal random variables with

$$E(\log b_i) = \mu_i, \quad Var(\log b_i) = \sigma_i^2.$$

where $c_j^k, j = 1,2,\ldots,n$ are random variables with known means for all values of k. It is given [Hines and Montgomory (1990)] that

$$E(b_i) = \exp\left(\mu_i + \frac{1}{2}\sigma_i^2\right)$$

$$Var(b_i) = [\exp(2\mu_i + \sigma_i^2)][\exp(\sigma_i^2 - 1)].$$

For simplicity, $a_{ij}, i = 1,2,\ldots,m,\ j = 1,2,\ldots,n,$ are assumed to be deterministic constants. The pdf of the i^{th} random variable $b_i, i = 1,2,\ldots,m,$ is given by

$$f_i(b_i) = \frac{1}{\sqrt{2\pi}\sigma_i b_i} \exp\left[-\frac{1}{2}\left(\frac{\log b_i - \mu_i}{\sigma_i}\right)^2\right], 0 < b_i < +\infty \qquad (4.3.18)$$

where $\qquad \sigma_i > 0, \log b_i = \ln b_i.$

Let

$$y_i = \sum_{j=1}^{n} a_{ij} x_j,\ i = 1,2,\ldots,m$$

where $y_i \geq 0$. Now, the joint probability constraints (4.3.2) can be written as

$$\Pr(b_1 \geq y_1, b_2 \geq y_2, \ldots, b_m \geq y_m) \geq 1 - \alpha.$$

Since $b_i, i = 1,2,\ldots,m,$ are independent random variables, the above joint constraints can be expressed as

$$\prod_{i=1}^{m} \Pr(b_i \geq y_i) \geq 1 - \alpha$$

where

$$\Pr(b_i \geq y_i) = \Pr\left(\frac{\log b_i - \mu_i}{\sigma_i} \geq \frac{\log y_i - \mu_i}{\sigma_i}\right),\ i = 1,2,\ldots,m$$

which can be simplified to

$$\Pr(b_i \geq y_i) = 1 - \Phi\left(\frac{\log y_i - \mu_i}{\sigma_i}\right), \; i = 1,2,\ldots,m.$$

Hence the joint constraints can be simplified to

$$\prod_{i=1}^{m}\left[1 - \Phi\left(\frac{\log y_i - \mu_i}{\sigma_i}\right)\right] \geq 1 - \alpha. \tag{4.3.19}$$

The equivalent multi-objective deterministic model of the probabilistic problem (4.3.1)-(4.3.3) can be stated as the E-model below:

$$\left.\begin{aligned}
& Max{:}Z_k = \sum_{j=1}^{n} E\left[c_j^k\right]x_j, \; k = 1,2,\ldots,K \\
& \text{Subject to} \\
& \prod_{i=1}^{m}\left[1 - \Phi\left(\frac{\log y_i - \mu_i}{\sigma_i}\right)\right] \geq 1 - \alpha. \\
& \sum_{j=1}^{n} a_{ij}x_j - y_i = 0, \; i = 1,2,\ldots,m \\
& y_i \geq 0, \; i = 1,2,\ldots,m. \\
& x_j \geq 0, \; j = 1,2,\ldots,n.
\end{aligned}\right\} \tag{4.3.19}$$

(v) When a_{ij} are log-normal random variables

It is assumed that $a_{ij}, i = 1,2,...,m, j = 1,2,...,n$, are independent log-normal random variables with

$$E(\log a_{ij}) = \mu_{ij},$$

$$Var(\log a_{ij}) = \sigma_{ij}^2, \; i = 1,2,...,m, \;\; j = 1,2,...,m$$

where $b_i, i = 1,2,...,m$ are deterministic constants, and c_j and $c_j^k, j = 1,2,...,n$ are random variables with known means for all values of k. Then [Hines and Montgomory (1990)]

$$E(a_{ij}) = \exp\left(\mu_{ij} + \frac{1}{2}\sigma_{ij}^2\right)$$

$$Var(a_{ij}) = [\exp(2\mu_{ij} + \sigma_{ij}^2)][\exp(\sigma_{ij}^2) - 1]$$

The pdf of the random variable $a_{ij}, i = 1,2,...,m, j = 1,2,...,n,$ is given by

$$f_{ij}(a_{ij}) = \frac{1}{\sqrt{2\pi}\sigma_{ij}a_{ij}} \exp\left[-\frac{1}{2}\left(\frac{\log a_{ij} - \mu_{ij}}{\sigma_{ij}}\right)^2\right], \; 0 < a_{ij} < +\infty \tag{4.3.20}$$

where $\sigma_{ij} > 0$ and $\log a_{ij} = \ln a_{ij}$.

Now, using the geometric inequality, we can write the joint probability constraints (4.3.2) as

$$\Pr\left(n\left(\prod_{j=1}^{n} a_{1j}x_j\right)^{1/n} \le b_1, n\left(\prod_{j=1}^{n} a_{2j}x_j\right)^{1/n} \le b_2, \ldots, n\left(\prod_{j=1}^{n} a_{mj}x_j\right)^{1/n} \le b_m, \right) \ge 1-\alpha \tag{4.3.21}$$

which can be simplified to

$$\prod_{i=1}^{m} \Pr\left(n\log n + \sum_{j=1}^{n} \log a_{ij} + \sum_{j=1}^{n} \log x_j \le n\log b_i \right) \ge 1-\alpha,$$

Let

$$y_i = \sum_{j=1}^{n} \log a_{ij}x_j,\ i = 1,2,\ldots,m$$

where y_i is a normal random variable with

$$E(y_i) = M_i = \sum_{j=1}^{n} \mu_{ij},\ \ i = 1,2,\ldots,m$$

$$Var(y_i) = S_i^2 = \sum_{j=1}^{n} \sigma_{ij}^2,\ i = 1,2,\ldots,m.$$

It is assumed that $y_i, i = 1,2,\dots,m$ are independent normal random variables. The joint constraints can be expressed as

$$\prod_{i=1}^{m} \Pr\left(y_i \le n\log(b_i / n) - \sum_{j=1}^{n} \log x_j \right) \ge 1 - \alpha \qquad (4.3.22)$$

where

$$\Pr\left(y_i \le n\log(b_i / n) - \sum_{j=1}^{n} \log x_j \right) = \Pr\left(\frac{y_i - M_i}{S_i} \le \frac{n\log(b_i / n) - \sum_{j=1}^{n} \log x_j - M_i}{S_i} \right)$$

which can be simplified to

$$\Pr\left(y_i \le n\log(b_i / n) - \sum_{j=1}^{n} \log x_j \right) = \Phi\left(\frac{n\log(b_i / n) - \sum_{j=1}^{n} \log x_j - M_i}{S_i} \right)$$

Hence the joint constraints can be simplified to

$$\prod_{i=1}^{m} \Phi\left(\frac{n\log(b_i / n) - \sum_{j=1}^{n} \log x_j - M_i}{S_i} \right) \ge 1 - \alpha \qquad (4.3.23)$$

Let

$$t_i = \Phi\left(\frac{n\log(b_i / n) - \sum_{j=1}^{n} \log x_j - M_i}{S_i} \right), \ i = 1,2,\dots,m.$$

where the t_i are positive numbers. The equivalent multi-objective deterministic model of the probabilistic problem (4.3.1)-(4.3.3) can be stated as the E-model given below:

$$\left.\begin{aligned} & Max{:}Z_k = \sum_{j=1}^{n} E\left[c_j^k\right] x_j,\ k = 1,2,\ldots,K \\ & \text{Subject to} \\ & \prod_{i=1}^{m} t_i \geq 1-\alpha \\ & \text{where} \quad t_i = \Phi\left(\frac{n\log(b_i/n) - \sum_{j=1}^{n}\log x_j - M_i}{S_i}\right) \\ & t_i \geq 0,\ i = 1,2,\ldots,m. \\ & x_j \geq 0,\ j = 1,2,\ldots,n. \end{aligned}\right\} \quad (4.3.24)$$

(*vi*) *When a_{ij} and b_i are log-normal random variables*

It is assumed that $a_{ij}; i = 1,2,\ldots,m; j = 1,2,\ldots,n$ and b_i, $i = 1,2,\ldots,m$, are independent log normal random variables with

$$E(\log b_i) = \mu_i,\ \ Var(\log b_i) = \sigma_i^2,\ \ i = 1,2,\ldots,m$$

$$E(\log a_{ij}) = \mu_{ij},\ \ Var(\log a_{ij}) = \sigma_{ij}^2,\ i = 1,2,\ldots,m,\ \ j = 1,2,\ldots,n$$

and c_j^k, $j = 1,2,\ldots,n$, are random variables with known means for all values of k. Then

$$E(b_i) = \exp\left(\mu_i + \frac{1}{2}\sigma_i^2\right)$$

$$Var(b_i) = [\exp(2\mu_i + \sigma_i^2)][\exp(\sigma_i^2) - 1]$$

$$E(a_{ij}) = \exp\left(\mu_{ij} + \frac{1}{2}\sigma_{ij}^2\right)$$

$$Var(a_{ij}) = [\exp(2\mu_{ij} + \sigma_{ij}^2)][\exp(\sigma_{ij}^2) - 1]$$

The pdf of the i^{th} random variable $b_i, i = 1,2,\ldots,m$, is given by

$$f_i(b_i) = \frac{1}{\sqrt{2\pi}\sigma_i b_i}\exp\left[-\frac{1}{2}\left(\frac{\log b_i - \mu_i}{\sigma_i}\right)^2\right], \ 0 < b_i < +\infty \tag{4.3.25}$$

where $\sigma_i > 0$ and $\log b_i = \ln b_i$.

Similarly, the pdf of the random variable a_{ij} is given by

$$f_{ij}(a_{ij}) = \frac{1}{\sqrt{2\pi}\sigma_{ij} a_{ij}}\exp\left[-\frac{1}{2}\left(\frac{\log a_{ij} - \mu_{ij}}{\sigma_{ij}}\right)^2\right], \ 0 < a_{ij} < +\infty \tag{4.3.26}$$

where $\sigma_{ij} > 0$ and $\log a_{ij} = \ln a_{ij}$.

Now, using the geometric inequality, we can write the joint probability constraints (4.3.2) as

$$\Pr\left(n\left(\prod_{j=1}^{n} a_{1j}x_j\right)^{1/n} \le b_1, n\left(\prod_{j=1}^{n} a_{2j}x_j\right)^{1/n} \le b_2, \ldots, n\left(\prod_{j=1}^{n} a_{mj}x_j\right)^{1\ n} \le b_m, \right) \ge 1-\alpha \tag{4.3.27}$$

which can be simplified to

$$\prod_{i=1}^{m} \Pr\left(\log n + (1/n)\sum_{j=1}^{n} \log(a_{ij}x_j) \le \log b_i \right) \ge 1-\alpha,$$

which can be written as

$$\prod_{i=1}^{m} \Pr\left(\sum_{j=1}^{n} \log a_{ij} - n\log b_i \le -\log(n^n) - \sum_{j=1}^{n} \log x_j \right) \ge 1-\alpha.$$

Let

$$y_i = \sum_{j=1}^{n} \log a_{ij} - n\log b_i,\ i = 1,2,\ldots,m. \tag{4.3.28}$$

where y_i is a normal random variable with

$$E(y_i) = M_i = \sum_{j=1}^{n} \mu_{ij} - n\mu_i,\ i = 1,2,\ldots,m$$

$$Var(y_i) = S_i^2 = n^2\sigma_i^2 + \sum_{j=1}^{n}\sigma_{ij}^2,\ i = 1,2,\ldots,m.$$

It is assumed that the $y_i, i = 1,2,\ldots,m,$ are independent normal random variables. The joint constraints can be expressed as

$$\prod_{i=1}^{m}\Pr\left(y_i \le -\log(n^n) - \sum_{j=1}^{n}\log x_j\right) \ge 1-\alpha \qquad (4.3.29)$$

where

$$\Pr\left(y_i \le -\log(n^n) - \sum_{j=1}^{n}\log x_j\right) = \Pr\left(\frac{y_i - M_i}{S_i} \le \frac{-\log(n^n) - \sum_{j=1}^{n}\log x_j - M_i}{S_i}\right)$$

which can be simplified to

$$\Pr\left(y_i \le -\log(n^n) - \sum_{j=1}^{n}\log x_j\right) = \Phi\left(\frac{-\log(n^n) - \sum_{j=1}^{n}\log x_j - M_i}{S_i}\right)$$

Hence the joint constraints can be simplified to

$$\prod\left[1-\Phi\left(\frac{\log(n^n) + \sum_{j=1}^{n}\log x_j + M_i}{S_i}\right)\right] \ge 1-\alpha. \qquad (4.3.30)$$

Let

$$t_i = 1 - \Phi\left(\frac{\log(n^n) + \sum_{j=1}^{n} \log x_j + M_i}{S_i}\right), \quad i = 1,2,\ldots,m.$$

where the t_i are positive numbers. The equivalent multi-objective deterministic model of the probabilistic problem (4.3.1)-(4.3.3) can be stated as the E-model given below:

$$\left.\begin{aligned}
&Max{:}Z_k = \sum_{j=1}^{n} E\left[c_j^k\right] x_j, \; k = 1,2,\ldots,K \\
&\text{Subject to} \\
&\qquad \prod_{i=1}^{m} t_i \geq 1 - \alpha \\
&\qquad t_i = 1 - \Phi\left(\frac{\log(n^n) + \sum_{j=1}^{n} \log x_j + M_i}{S_i}\right) \\
&\qquad t_i \geq 0, \; i = 1,2,\ldots,m. \\
&\qquad x_j \geq 0, \; j = 1,2,\ldots,n.
\end{aligned}\right\} \quad (4.3.31)$$

4.5 Numerical example

We now consider the following multi-objective probabilistic linear programming problem involving normal random variables:

$$Max\, f_1 = 2x_1 + 3x_2 + x_3 \tag{4.4.1}$$

$$Max\, f_2 = 5x_1 + 2x_2 + 4x_3 \tag{4.4.2}$$

Subject to

$$\Pr(x_1 + x_2 + 2x_3 \geq b_1, x_1 + x_2 + x_3 \geq b_2, 3x_1 + x_2 + x_3 \geq b_3) \geq 0.90 \tag{4.4.3}$$

$$x_1, x_2, x_3 \geq 0 \tag{4.4.4}$$

Where b_1, b_2 and b_3 are independent normal random variables with known distributions. It is given that $E(b_1) = 10$, $E(b_2) = 4$, $E(b_3) = 20$, $Var(b_1) = 4$, $Var(b_2) = 1$ and $Var(b_3) = 9$. Using article [4.4(i)] we can present the deterministic model of the probabilistic problem as follows:

$$Max\, f_1 = 2x_1 + 3x_2 + x_3 \tag{4.4.5}$$

$$Max\, f_2 = 5x_1 + 2x_2 + 4x_3 \tag{4.4.6}$$

Subject to

$$\left[1-\Phi\left(\frac{y_1-10}{2}\right)\right]\left[1-\Phi\left(\frac{y_2-4}{1}\right)\right]\left[1-\Phi\left(\frac{y_3-20}{3}\right)\right]\geq 0.90 \tag{4.4.7}$$

$$x_1+x_2+2x_3=y_1 \tag{4.4.8}$$

$$x_1+x_2+x_3=y_2 \tag{4.4.9}$$

$$3x_1+x_2+x_3=y_3 \tag{4.4.10}$$

$$x_1,x_2,x_3,y_1,y_2,y_3\geq 0. \tag{4.4.11}$$

The problem is solved using the fuzzy programming technique and the solution is obtained as follows.

The ideal solution for $f_1(x)$ is $X^{(1)}=(0, 2.717751, 0)$ and the ideal solution for $f_2(x)$ is $X^{(2)}=(2.717550, 0, 0)$ where $f_1(x)=8.153254$ and $f_2(x)=13.58775$. Using the max-min operators formulate a crisp model as follows

$$Min\ \lambda \tag{4.4.12}$$

Subject to

$$2x_1+3x_2+x_3+2.718154\lambda\geq 8.153254 \tag{4.4.13}$$

$$5x_1+2x_2+4x_3+8.152248\lambda\geq 13.58775 \tag{4.4.14}$$

$$\left[1-\Phi\left(\frac{y_1-10}{2}\right)\right]\left[1-\Phi\left(\frac{y_2-4}{1}\right)\right]\left[1-\Phi\left(\frac{y_3-20}{3}\right)\right] \geq 0.90 \tag{4.4.15}$$

$$x_1 + x_2 + 2x_3 = y_1 \tag{4.4.16}$$

$$x_1 + x_2 + x_3 = y_2 \tag{4.4.17}$$

$$3x_1 + x_2 + x_3 = y_3 \tag{4.4.18}$$

$$x_1, x_2, x_3, y_1, y_2, y_3 \geq 0. \tag{4.4.19}$$

This problem is solved using an NLP package and the following optimal compromise solution is obtained: $x_1^* = 1.358885$, $x_2^* = 1.358862$, $x_3^* = 0.0$ and $\lambda^* = 0.4999355$, where $f_1^* = 6.794356$ and $f_2^* = 9.512149$.

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